Binary Search Trees

K08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού

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Search

- Searching for a specific value within a large collection is fundamental
- We want this to be efficient even if we have billions of values!
- So far we have seens two basic search strategies:
  - **sequential** search: slow
  - **binary** search: fast
    - but only for **sorted** data

Sequential search

```
// Αναζητά τον ακέραιο target στον πίνακα target. Επιστρέφει
// τη θέση του στοιχείου αν βρεθεί, διαφορετικά -1.

int sequential_search(int target, int array[], int size) {
  for (int i = 0; i < size; i++)
      if (array[i] == target)
          return i;

  return -1;
}
```

We already saw that the complexity is $O(n)$.

Binary search

```
// Αναζητά τον ακέραιο target στον __ταξινομημένο__ πίνακα target.
// Επιστρέφει τη θέση του στοιχείου αν βρεθεί, διαφορετικά -1.

int binary_search(int target, int array[], int size) {
    int low = 0;
    int high = size - 1;

    while (low <= high) {
        int middle = (low + high) / 2;

        if (target == array[middle])
            return middle;              // βρέθηκε
        else if (target > array[middle])
            low = middle + 1;           // συνεχίζουμε στο πάνω μισο
        else
            high = middle - 1;          // συνεχίζουμε στο κάτω μισό
    }

    return -1;
}
```

Important: the array needs to be **sorted**

Binary search example

Seaching For | | Result | |

5	50	119	210	248	270	356	425	434	519	547	604	748	874	900	941
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

At each step the search space is cut in half.

Binary search example

```
def binarySearch(listData, value)
    low = 0
    high = len(listData) - 1
    while (low <= high)
        mid = (low + high) / 2
        if (listData[mid] == value):
            return mid
        elif (listData[mid] < value)
            low = mid + 1
        else:
            high = mid - 1
    return -1
```

Seaching For 119 Result 2 Element found

low 2 mid 2 high 2

5	50	119	210	248	270	356	425	434	519	547	604	748	874	900	941
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

At each step the search space is cut in half.

Complexity of binary search

- **Search space**: the elements remaining to search
  - those between `low` and `right`
- The size of the search space is **cut in half** at each step
  - After step i there are $\frac{n}{2^i}$ elements remaining
- We **stop** when $\frac{n}{2^i} < 1$
  - in other words when $n < 2^i$
  - or equivalently when $\log n < i$
- So we will do at most $\log n$ steps
  - complexity $O(\log n)$
  - **30 steps** for one **billion** elements

Conclusions

- Binary search is fundamental for efficient search
- But we need **sorted data**
- Maintaining a sorted array **after an insert** is hard
  - complexity?
- How can we keep data sorted **and simultaneously** allow efficient inserts?

Binary Search Trees (BST)

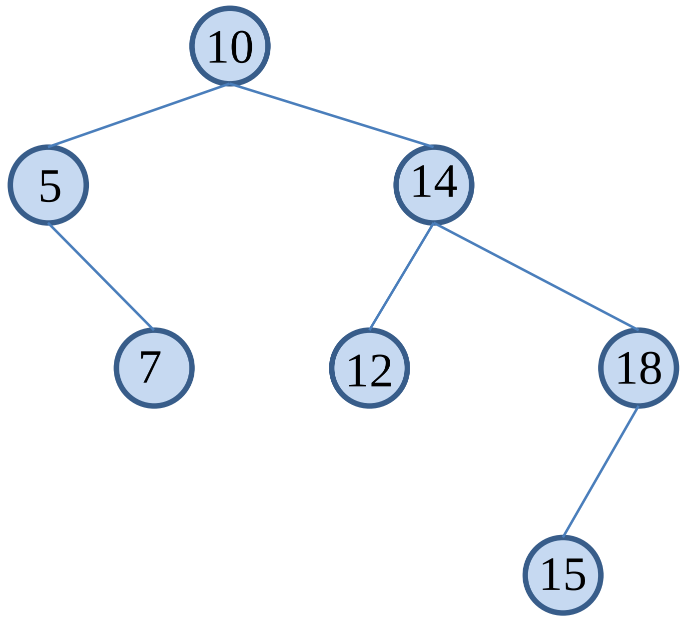
A **binary search tree** (δυαδικό δέντρο αναζήτησης) is a binary tree such that:

- every node is **larger** than all nodes on its **left subtree**
- every node is **smaller** than all nodes on its **right subtree**

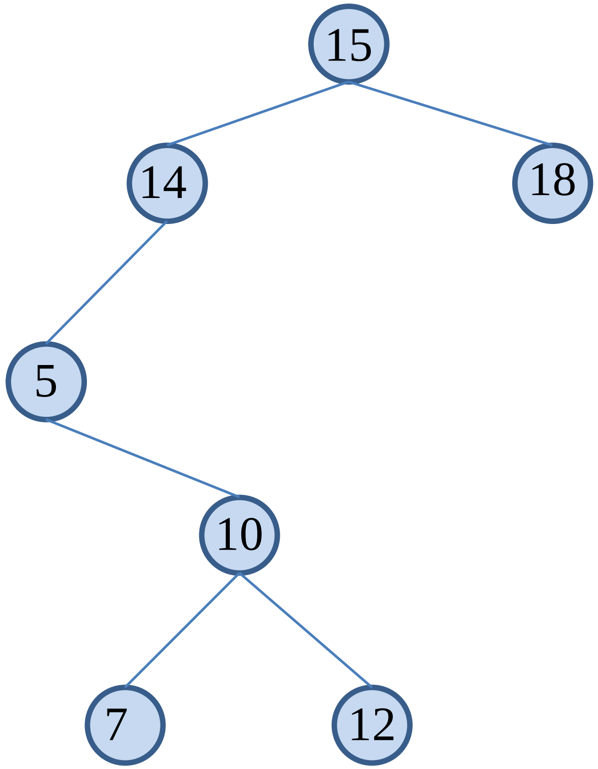
Note

- No value can appear **twice**
  (it would violate the definition)
- **Any** `compare` function can be used for ordering.
  (with some mathematical constraints, see the piazza post)

Example

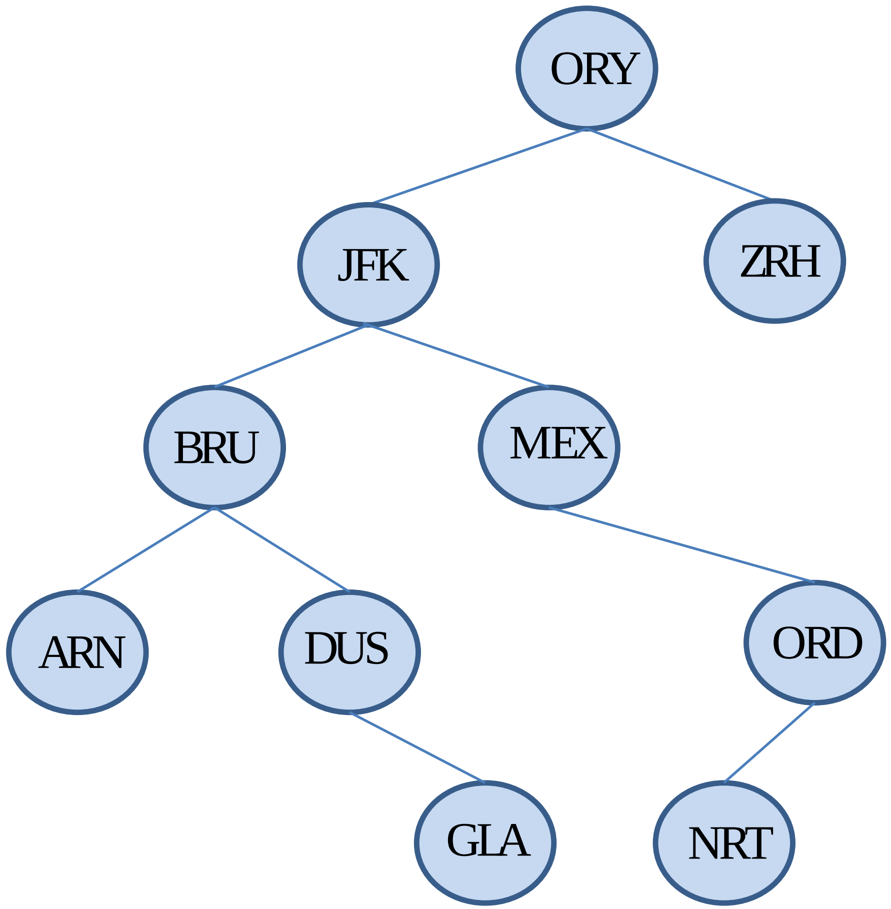


Example



A different tree with the **same values**!

Example



BST operations

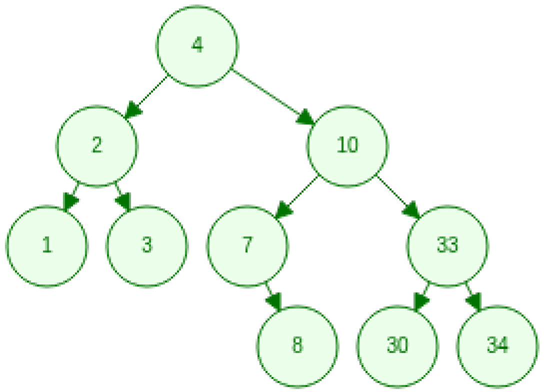
- Container operations
  - **Insert** / **Remove**
- **Search** for a given value
- **Ordered** traversal
  - Find **first** / **last**
  - Find **next** / **previous**
- So we can use BSTs to implement
  - **ADTMap** (we need search)
  - **ADTSet** (we need search and ordered traversal)

Search

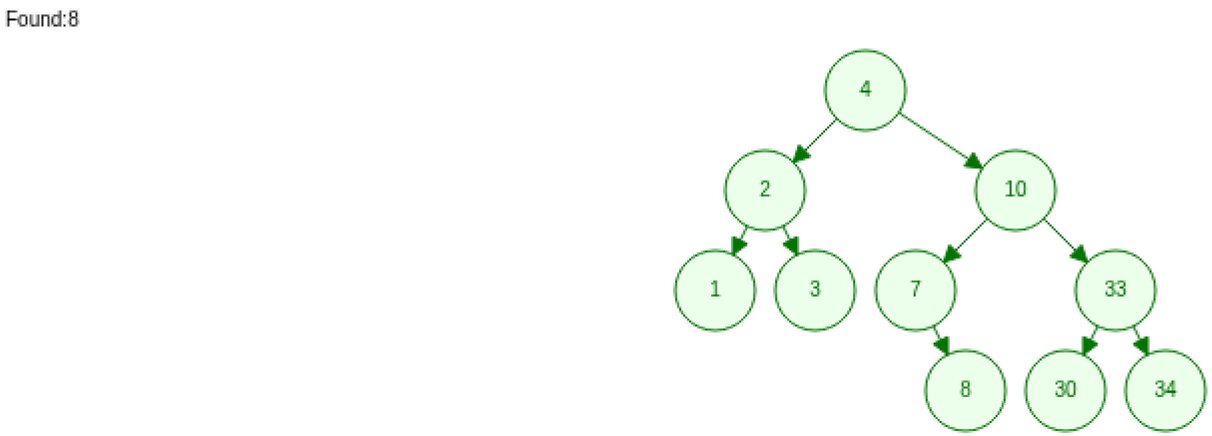
We perform the following procedure **starting at the root**

- If the tree is empty
  - `target` does not exist in the tree
- If `target` = `current_node`
  - Found!
- If `target` < `current_node`
  - continue in the **left subtree**
- If `target` > `current_node`
  - continue in the **right subtree**

Search example



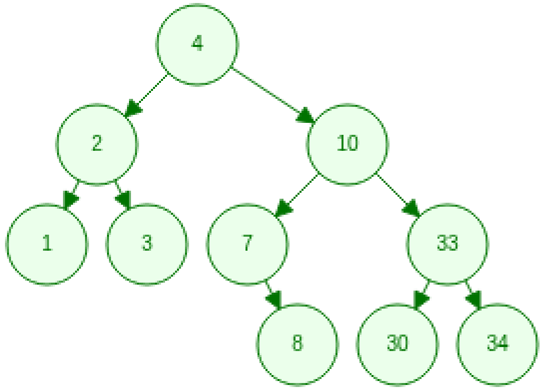
Search example



Searching for 8

Search example

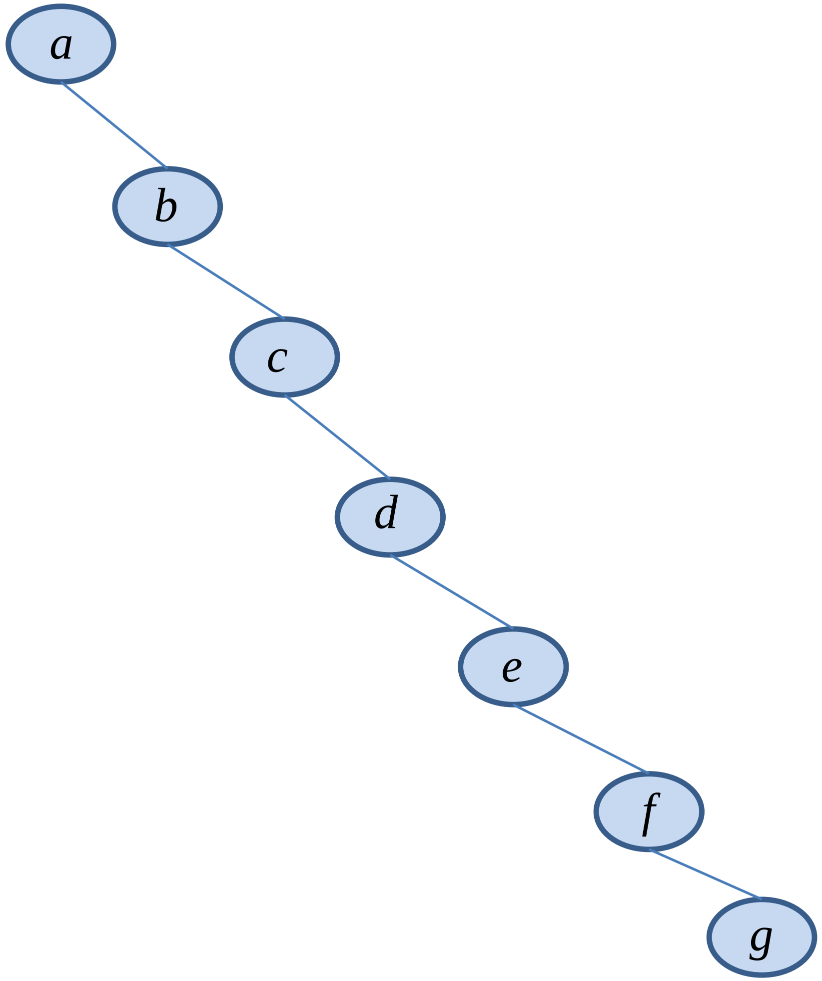
Found:8



Complexity of search

- How many steps will we make in the worst case?
  - We will traverse a path from the root to the tree
  - h steps max (the **height** of the tree)
- But how does h relate to n?
  - h = $O(n)$ in the worst case!
  - when the tree is essentially a degenerate “list”

Searching in this tree is slow



Complexity of search

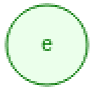
- This is a very common pattern in trees
  - Many operations are $O(h)$
  - Which means worst-case $O(n)$
- Unless we manage to **keep the tree short**!
  - We already saw this in **complete** trees, in which $h \leq \log n$
- Unfortunately maintaining a complete BST is not easy (why?)
  - But there are other methods to achieve the same result
    - AVL, B-Trees, etc
  - We will talk about them later

Inserting a new value

- Inserting a `value` is **very similar to search**
- We follow the same algorithm as if we were searching for `value`
  - If `value` is found we stop (no duplicates!)
  - If we reach an **empty subtree** insert `value` **there**

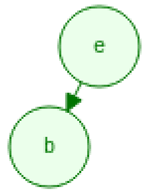
Insert example

Insert example



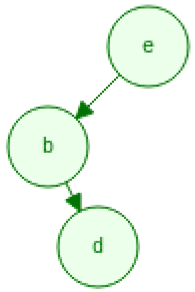
Inserting e

Insert example



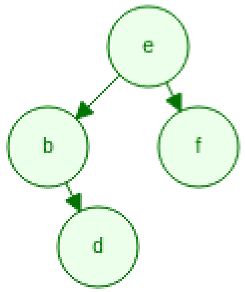
Inserting b

Insert example



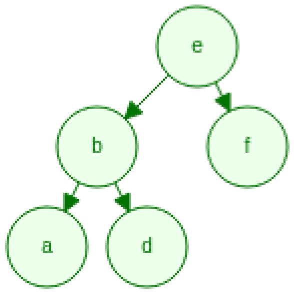
Inserting d

Insert example



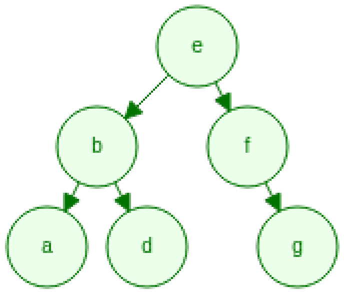
Inserting f

Insert example



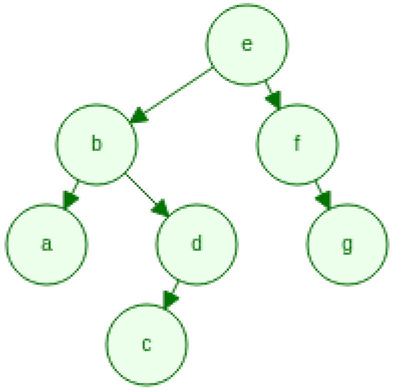
Inserting a

Insert example



Inserting g

Insert example



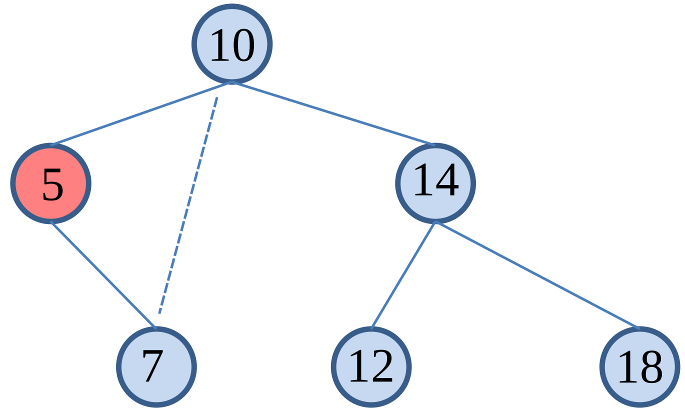
Inserting c

Complexity of insert

- Same as **search**
- $O(h)$
  - So $O(n)$ unless the tree is short

Deleting a value

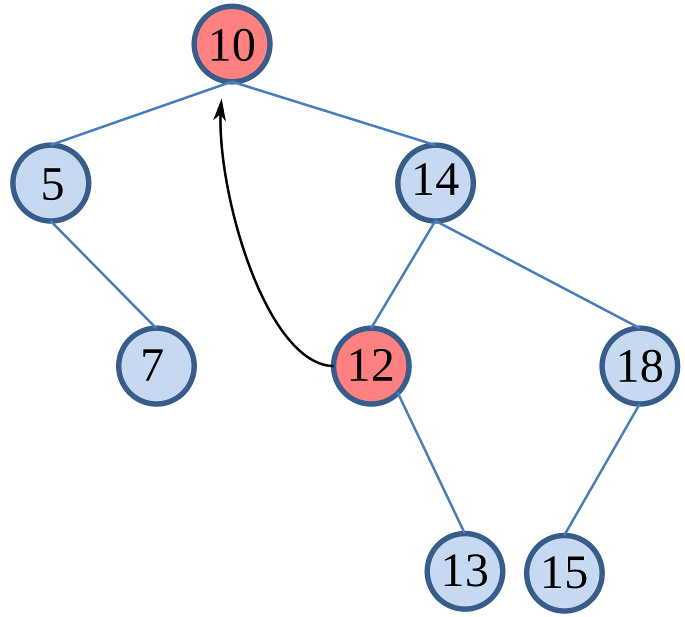
- We might want to delete **any node** in a BST
- Easy case: `node` has **as most 1 child**
- Connect the child directly to `node`'s **parent**
- BST property is preserved (why?)



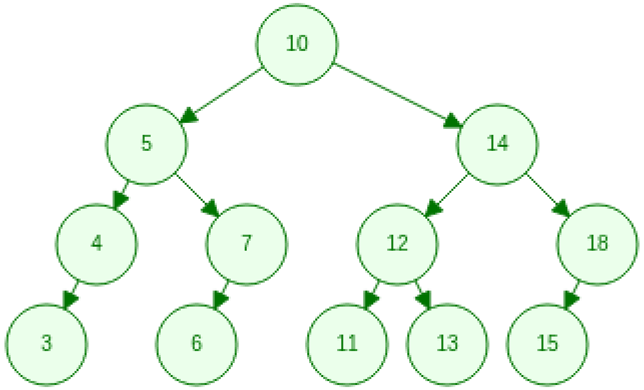
Deleting a value

- Hard case: `node` has **two children** (eg. 10)
- Find the `next` node in the order (eg. 12)
  - **left-most** node in the right sub-tree!

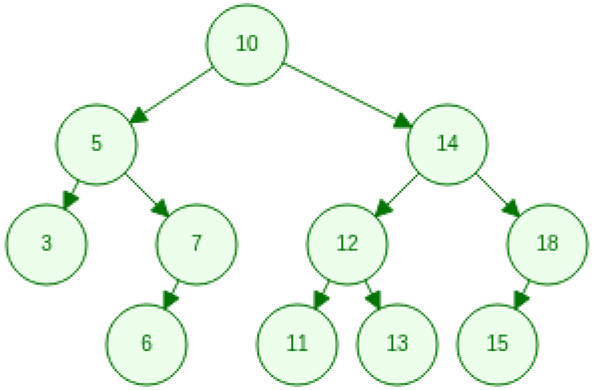
  (or equivalently the `previous` node)
- We can replace `node`'s value with `next`'s
  - this preserves the BST property (why?)
- And then delete `next`
  - This has to be an **easy** case (why?)



Delete example

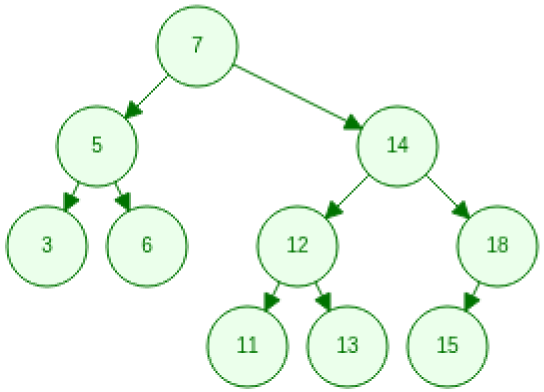


Delete example



Delete 4 (easy).

Delete example



Delete 10 (hard). Replace with 7 and it becomes easy.

Complexity of delete

- Finding the node to delete is $O(h)$
- Finding the `next` / `previous` is also $O(h)$

Ordered traversal: first/last

- How to find the **first** node?
  - simply follow left children
  - $O(h)$
  - same for **last**

Ordered traversal: next

- How to find the **next** of a given `node`?
- Easy case: the node has a right child
  - find the left-most node of the right subtree
  - we used this for **delete**!
- Hard case: no right-child, we need to go up!

Ordered traversal: next

General algorithm for any node.
Perform the following procedure **starting at the root**

```
// Ψευδοκώδικας, current_node είναι η ρίζα του τρέχοντος υποδέντρου,
// node είναι ο κόμβος του οποίου τον επόμενο ψάχνουμε.

find_next(current_node, node) {
    if (node == current_node) {
        // Ο target είναι η ρίζα του υποδέντρου, ο επόμενος είναι ο μ
        // του δεξιού υποδέντρου (αν είναι κενό τότε δεν υπάρχει επόμ
        return node_find_min(right_child);      // NULL αν δεν υπάρχε

    } else if (node > current_node)) {
        // Ο target είναι στο αριστερό υποδέντρο,
        // οπότε και ο προηγούμενός του είναι εκεί.
        return node_find_next(node->right, compare, target);

    } else {
        // Ο target είναι στο αριστερό υποδέντρο, ο επόμενός του μπορ
        // επίσης εκεί, αν όχι ο επόμενός του είναι ο ίδιος ο node.
        res = node_find_next(node->left, compare, target);
        return res != NULL ? res : node;
    }
}
```

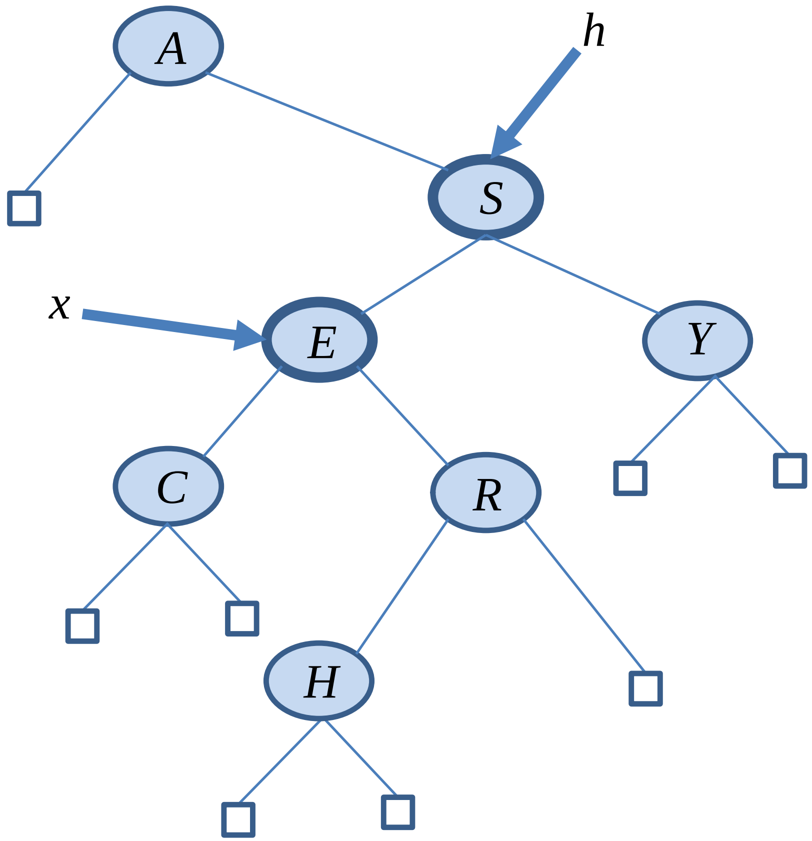
Complexity of next

- Similar to search, traversing the tree from the root to the leaves
  - so $O(h)$
- We can do it faster by keeping more structure
- We can keep a bidirectional list of all nodes in order
  - $O(1)$ to find next, no extra complexity to update
- More advanced: keep a **link to the parent**
  - Find the next by going **up** when needed
  - Can you find the algorithm?
  - Real-time complexity is still $O(h)$ if we traverse to the root
  - But what about amortized-time?

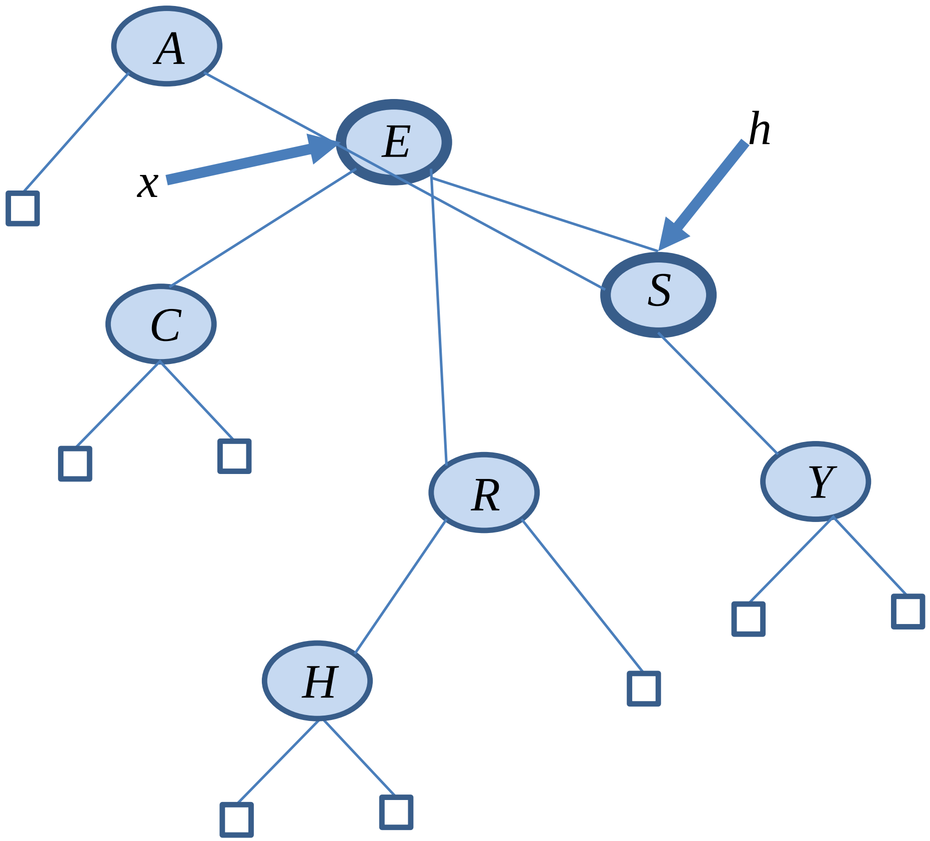
Rotations

- **Rotation (περιστροφή)** is a fundamental operation in BSTs
  - swaps the role of a **node and one of its children**
  - while still **preserving the BST property**
- **Right rotation**
  - swap a node h and its **left child** x
  - x becomes the root of the subtree
  - the **right** child of x becomes **left** child of h
  - h becomes a **right** child of x
- **Left rotation**
  - symmetric operation with **right** child

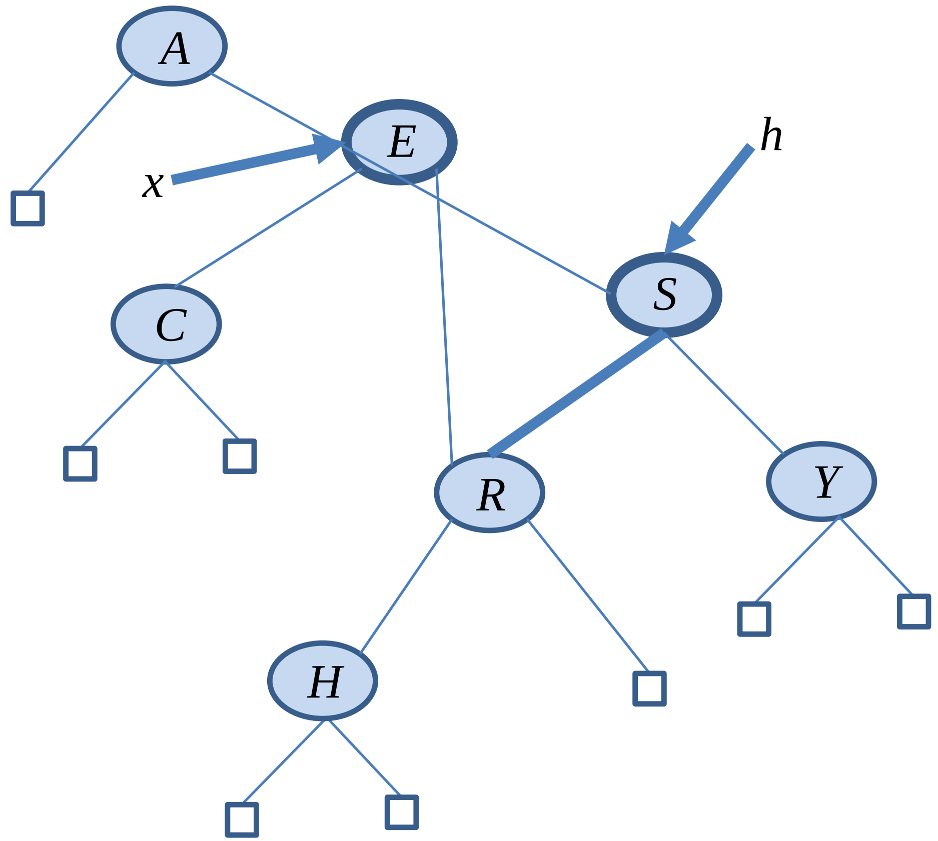
Example: right rotation



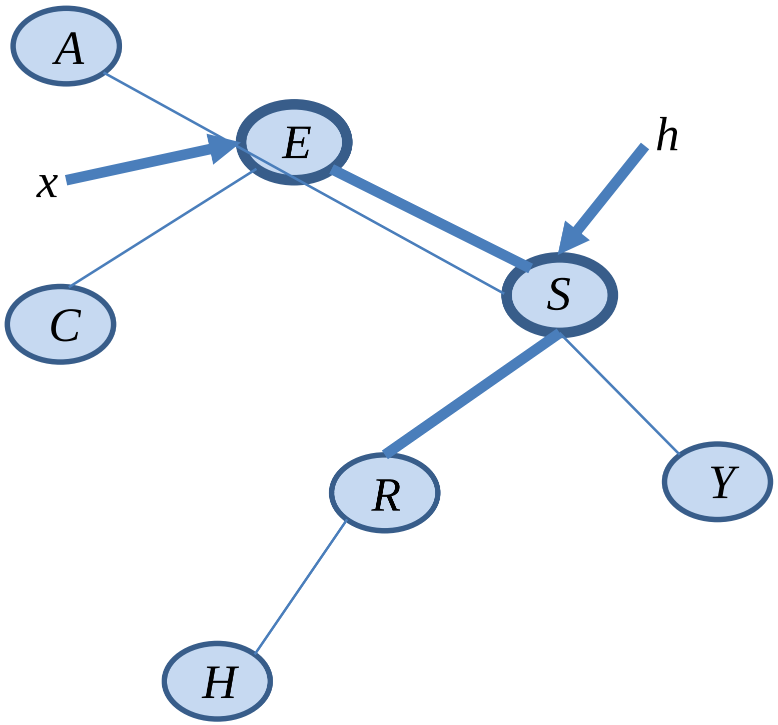
Example: right rotation



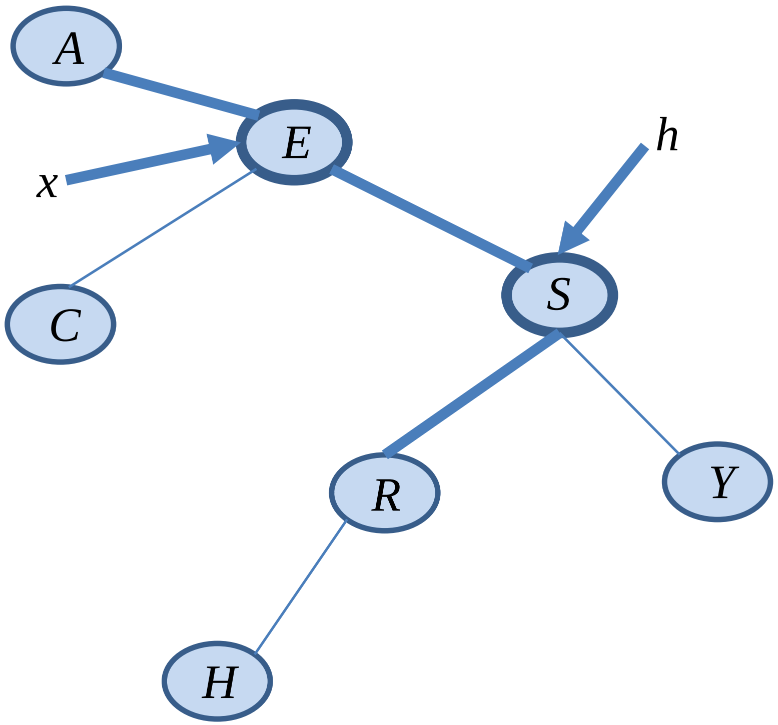
Example: right rotation



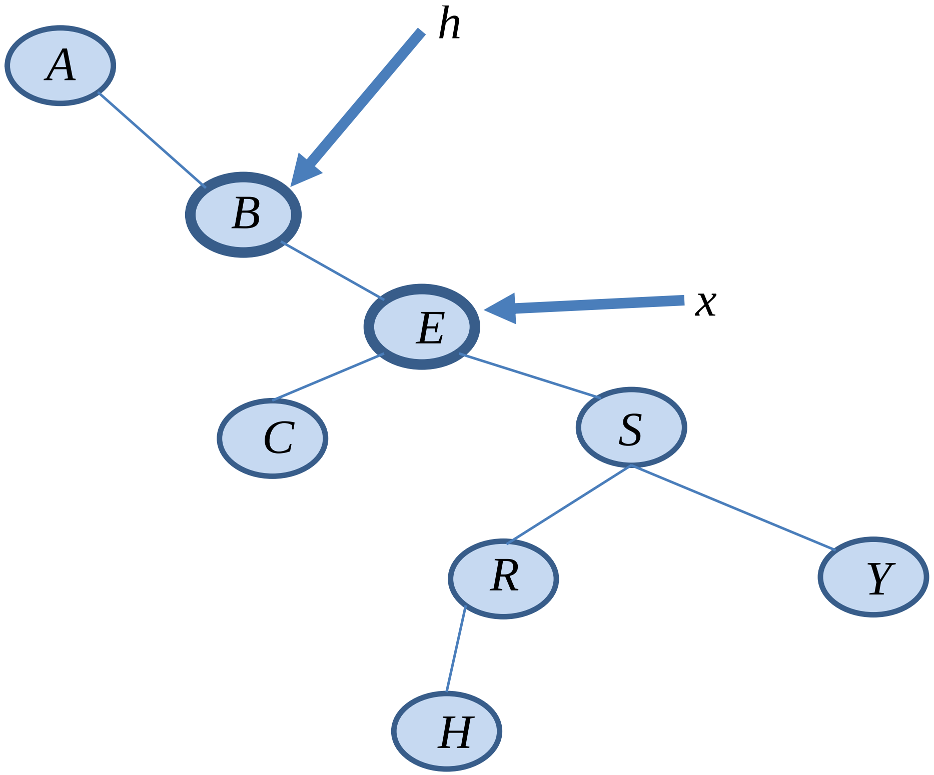
Example: right rotation



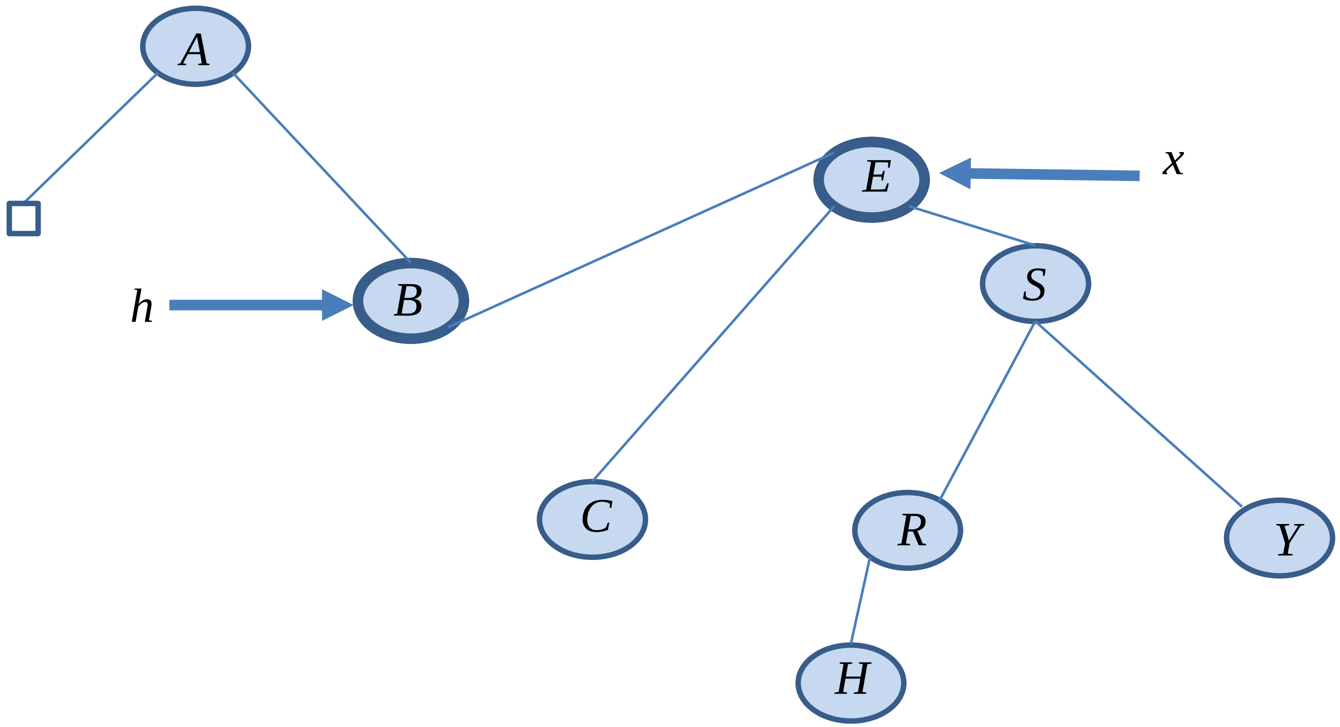
Example: right rotation



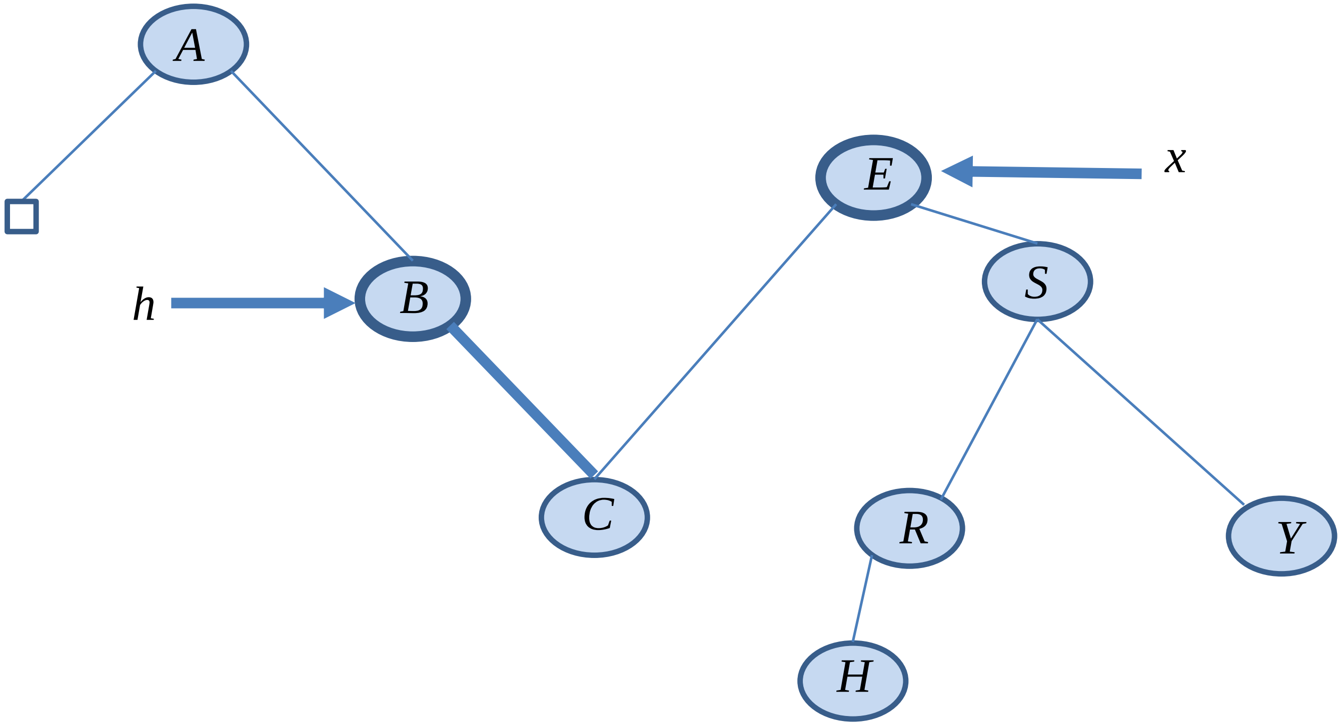
Example: left rotation



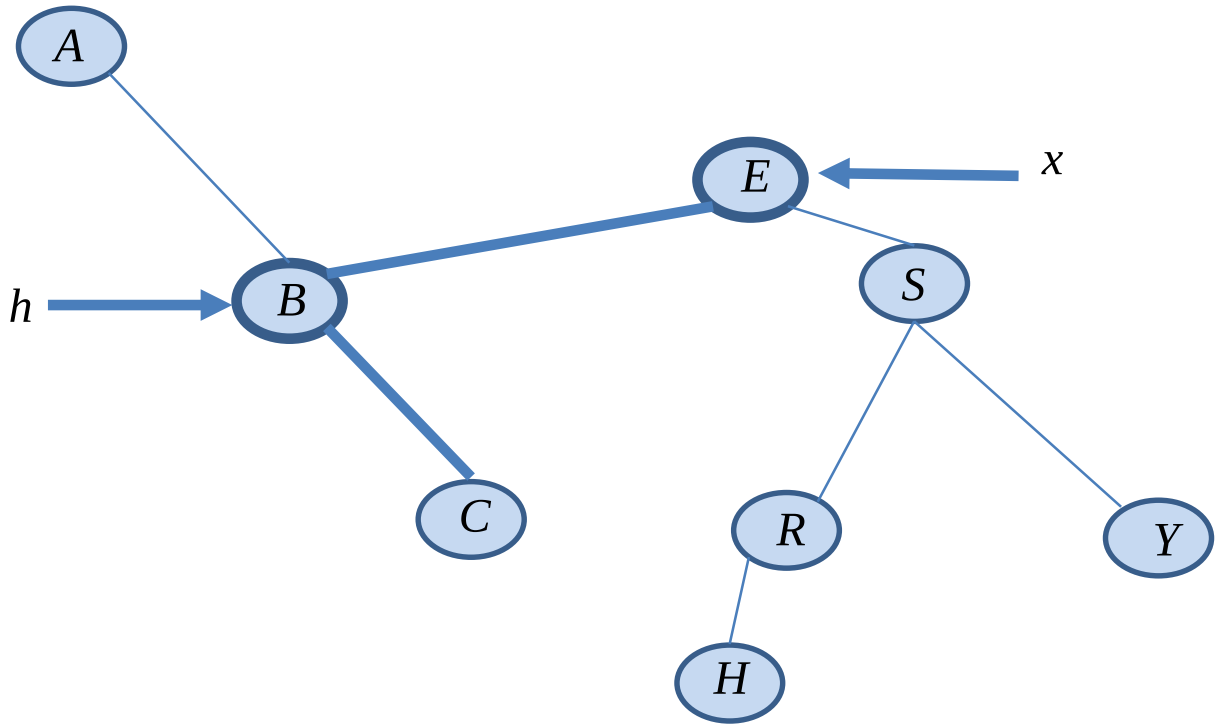
Example: left rotation



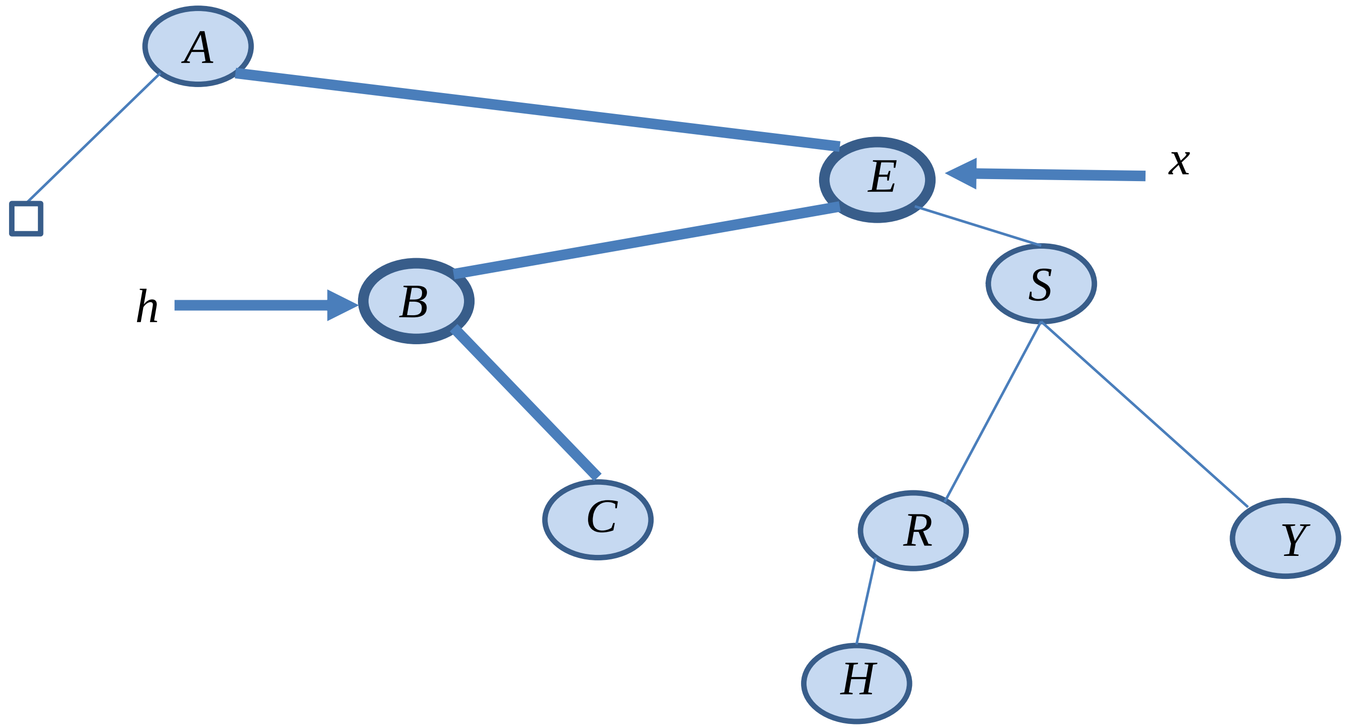
Example: left rotation



Example: left rotation



Example: left rotation



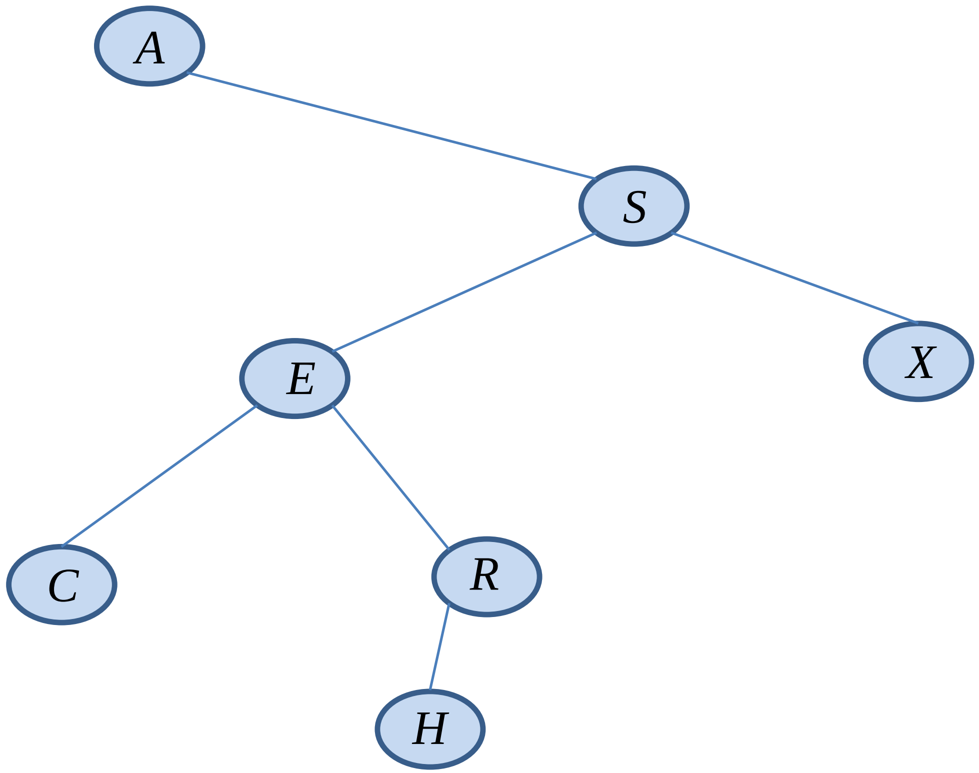
Complexity of rotation

- Only changing a few pointers
- No traversal of the tree!
- So $O(1)$

Root insertion

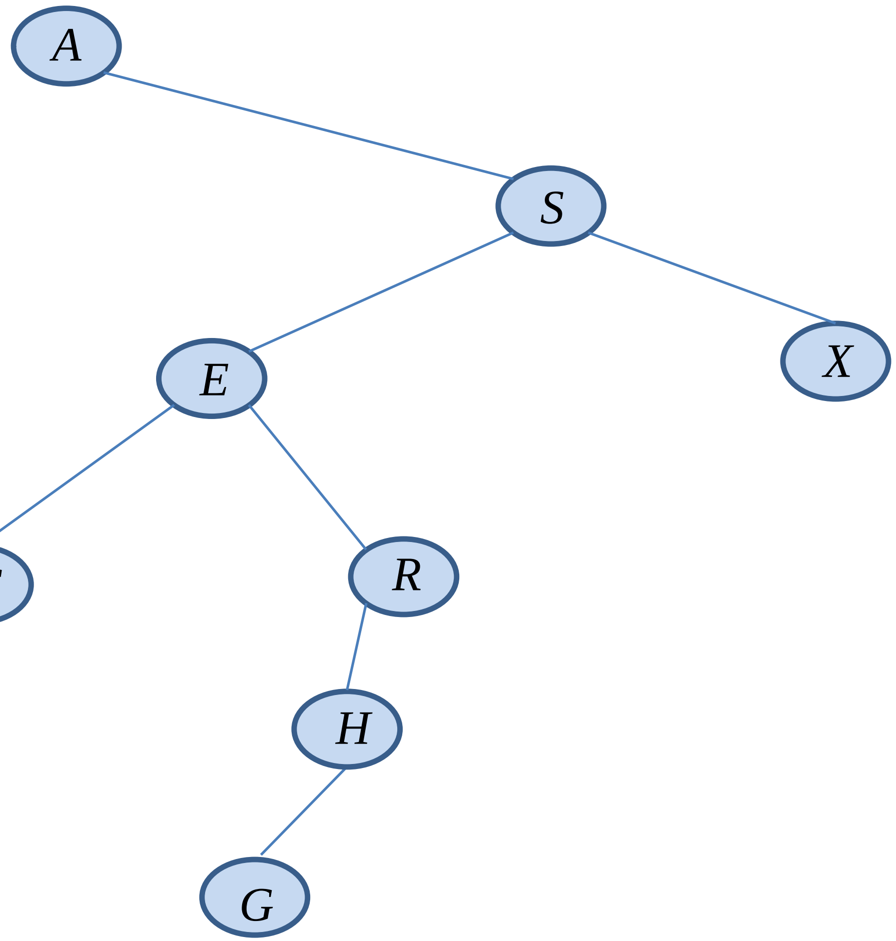
- Goal
  - insert a new element
  - place it **at the root** of the tree
- Simple **recursive** algorithm using **rotations**
  1. If empty: trivial
  2. **Recursively** insert in the **left/right subtree**
     - depending on whether the value is smaller than the root or not
     - after the recursive call finishes we have a **proper BST**
     - with the value as the **root** of the left/right **subtree**
  3. **Rotate** left or right
     - the value comes at the root!

Example: root insertion



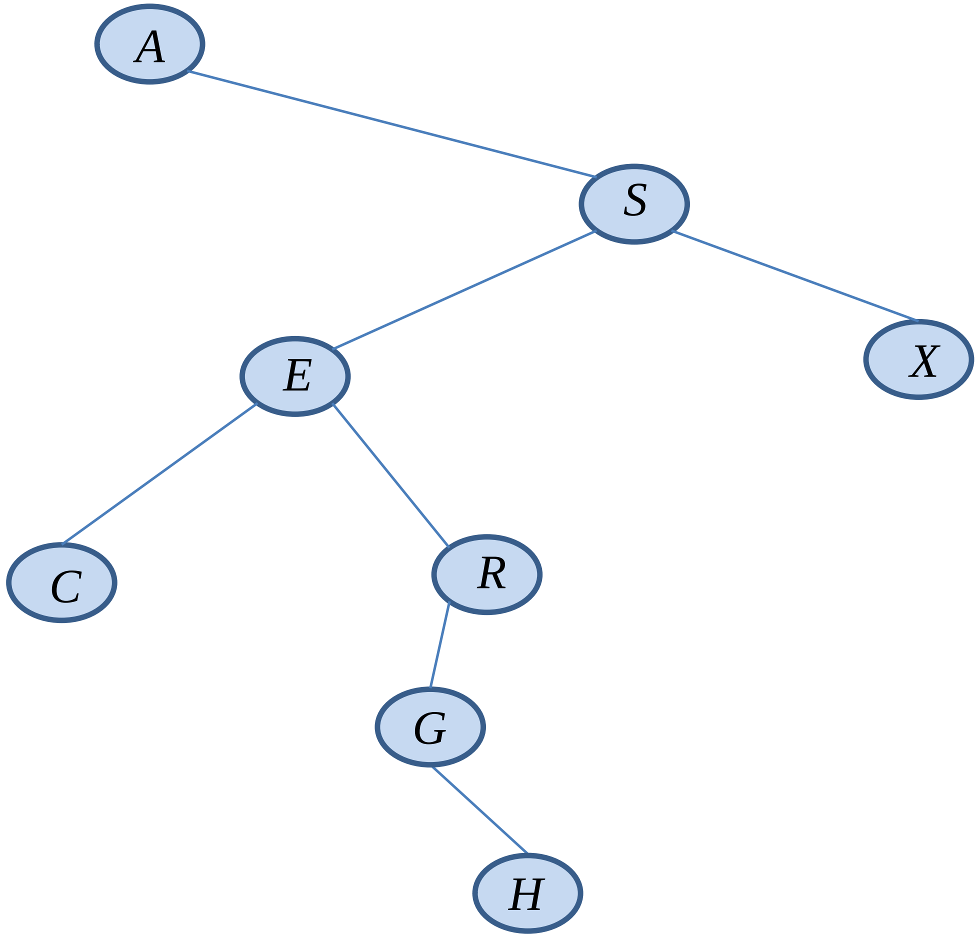
We are inserting G. The recursive algorithm is first called on the root A, then it makes **recursive calls** on the right subtree S, then on E, R, H, and finally a recursive call is made on the empty left subtree of H.

Example: root insertion



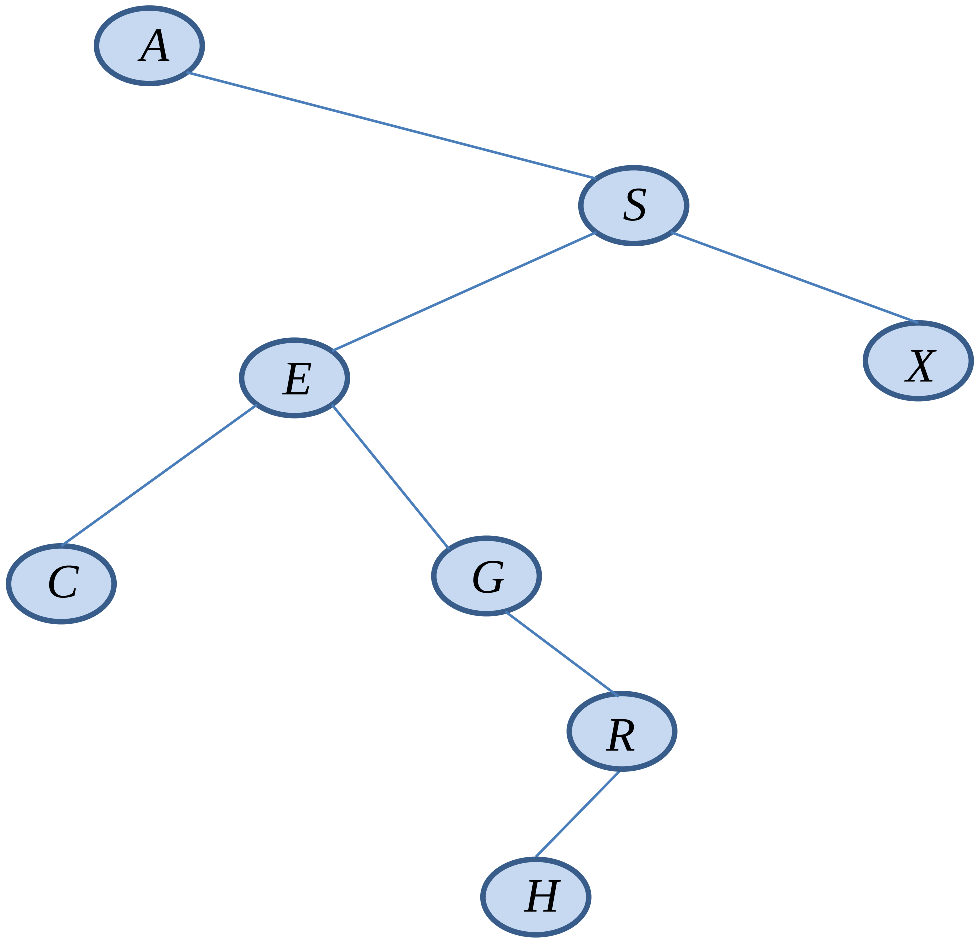
G is inserted in the empty left subtree of H.

Example: root insertion



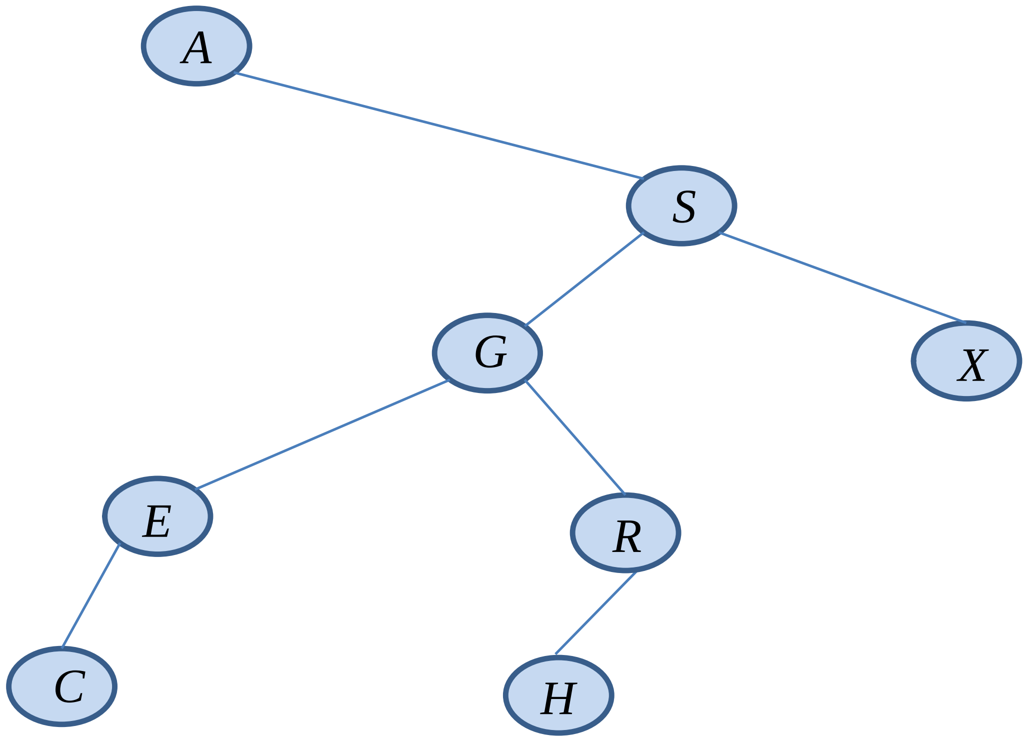
The call on H does a right rotation, G moves up.

Example: root insertion



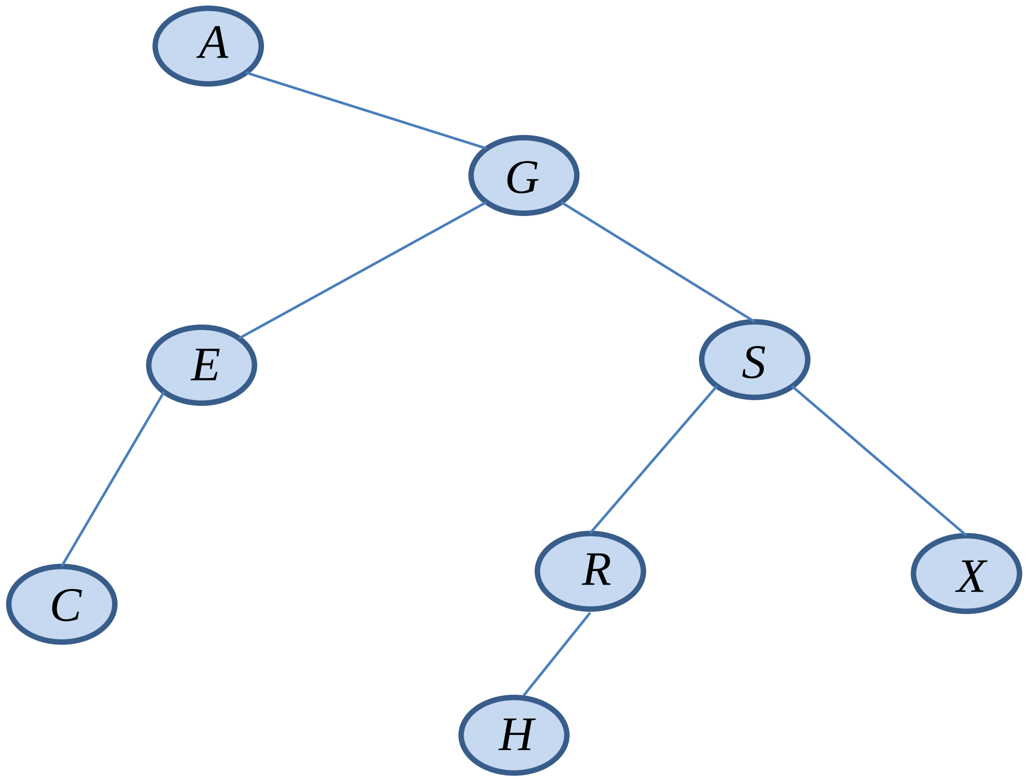
The call on R does a right rotation, G moves up.

Example: root insertion



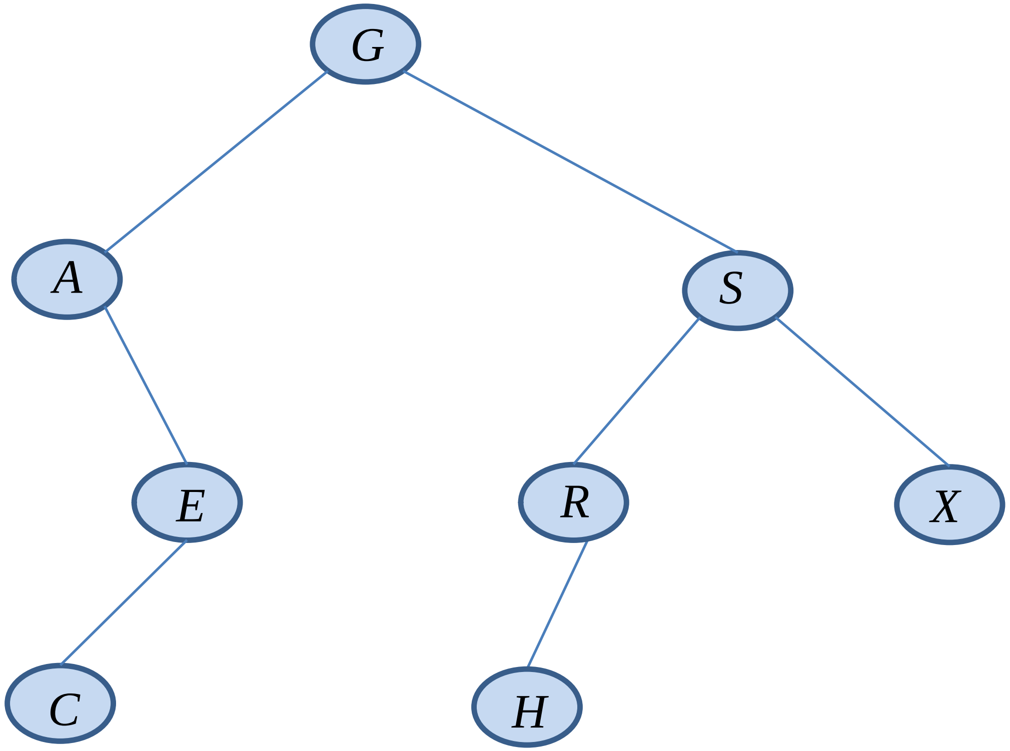
The call on E does a left rotation, G moves up.

Example: root insertion



The call on R does a right rotation, G moves up.

Example: root insertion



The call on A does a left rotation, G arrives at the root.

Complexity of root insertion

- The algorithm is similar to a normal insert
  - traversing the tree towards the leaves: $O(h)$
- With an **extra rotation** at every step
  - which is $O(1)$
- So still $O(h)$

Readings

- T. A. Standish. *Data Structures, Algorithms and Software Principles in C.*
  - Chapter 5. Sections 5.6 and 6.5.
  - Chapter 9. Section 9.7.
- R. Sedgewick. Αλγόριθμοι σε C.
  - Κεφ. 12.
- M.T. Goodrich, R. Tamassia and D. Mount. *Data Structures and Algorithms in C++*. 2nd edition.
  - Section 9.3 and 10.1