Recursion

K08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού

Under construction

Recursion

- Recursion is a **fundamental concept** of Computer Science.
- It usually help us to write simple and elegant solutions to programming problems.
- You will learn to program recursively by working with many examples to develop your skills.

Recursive Programs

A **recursive program** is one that calls itself in order to obtain a solution to a problem.

The reason that it calls itself is to compute a solution to a **subproblem** that has the following properties:

- The subproblem is **smaller** than the problem to be solved.
- The subproblem can be solved **directly (as a base case)** or **recursively by making a recursive call**.
- The subproblem's solution can be **combined** with solutions to other subproblems to obtain a solution to the overall problem.

Example

- Let us consider a simple program to add up all the squares of integers from m to n.
- An **iterative function** to do this is the following:

```
int SumSquares(int m, int n) {
  int i, sum;
  sum=0;
  for (i = m; i <= n; i++) sum += i*i;
  return sum;
}
```

Recursive Sum of Squares

```
int SumSquares(int m, int n) {
  if (m < n) {
    return m*m + SumSquares(m+1, n);
  }
  else {
    return m*m;
  }
}
```

Comments

- In the case that the range m:n contains more than one number, the solution to the problem can be found by adding (a) the solution to the smaller subproblem of summing the squares in the range m+1:n and (b) the solution to the subproblem of finding the square of m. (a) is then solved in the same way (recursion).

- We stop when we reach the **base case** that occurs when the range m:n contains just one number, in which case m==n.

- This recursive solution can be called **“going-up” recursion** since the successive ranges are m+1:n, m+2:n etc.

Going-Down Recursion

```
int SumSquares(int m, int n) {
  if (m < n) {
    return SumSquares(m, n-1) + n*n;
  }
  else {
    return  n*n;
  }
}
```

Recursion Combining Two Half-Solutions

```
int SumSquares(int m, int n) {
  int middle;
  if (m == n) {
    return m*m;
  }
  else {
    middle = (m + n) / 2;
  return;
  SumSquares(m,middle) + SumSquares(middle + 1,n);
  }
}
```

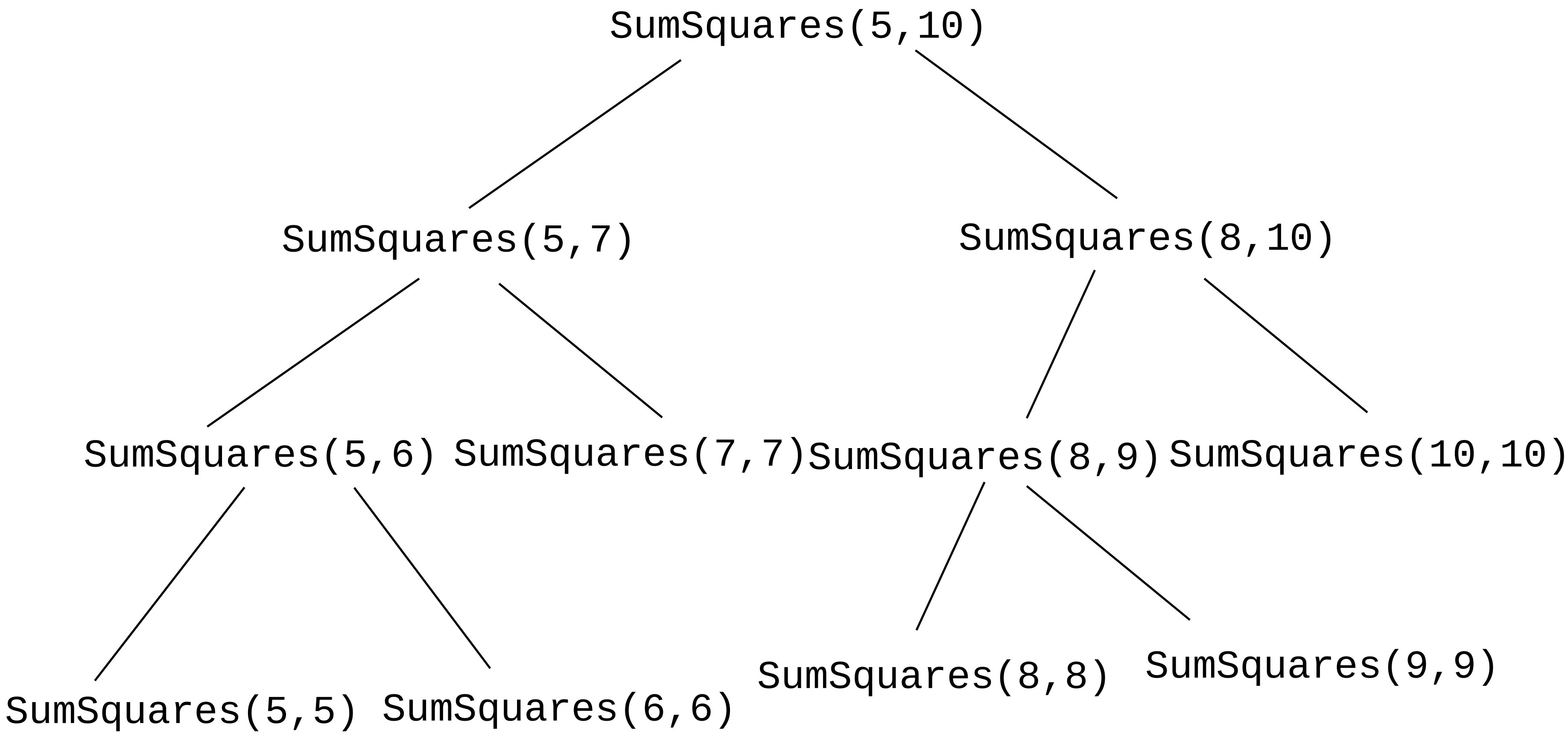
Comments

- The **recursion** here says that the sum of the squares of the integers in the range m:n can be obtained by adding the sum of the squares of the left half range, m:middle, to the sum of the squares of the right half range, middle+1:n.
- We stop when we reach the **base case** that occurs when the range contains just one number, in which case m==n.
- The middle is computed by using **integer division** (operator /) which keeps the quotient and throws away the remainder.

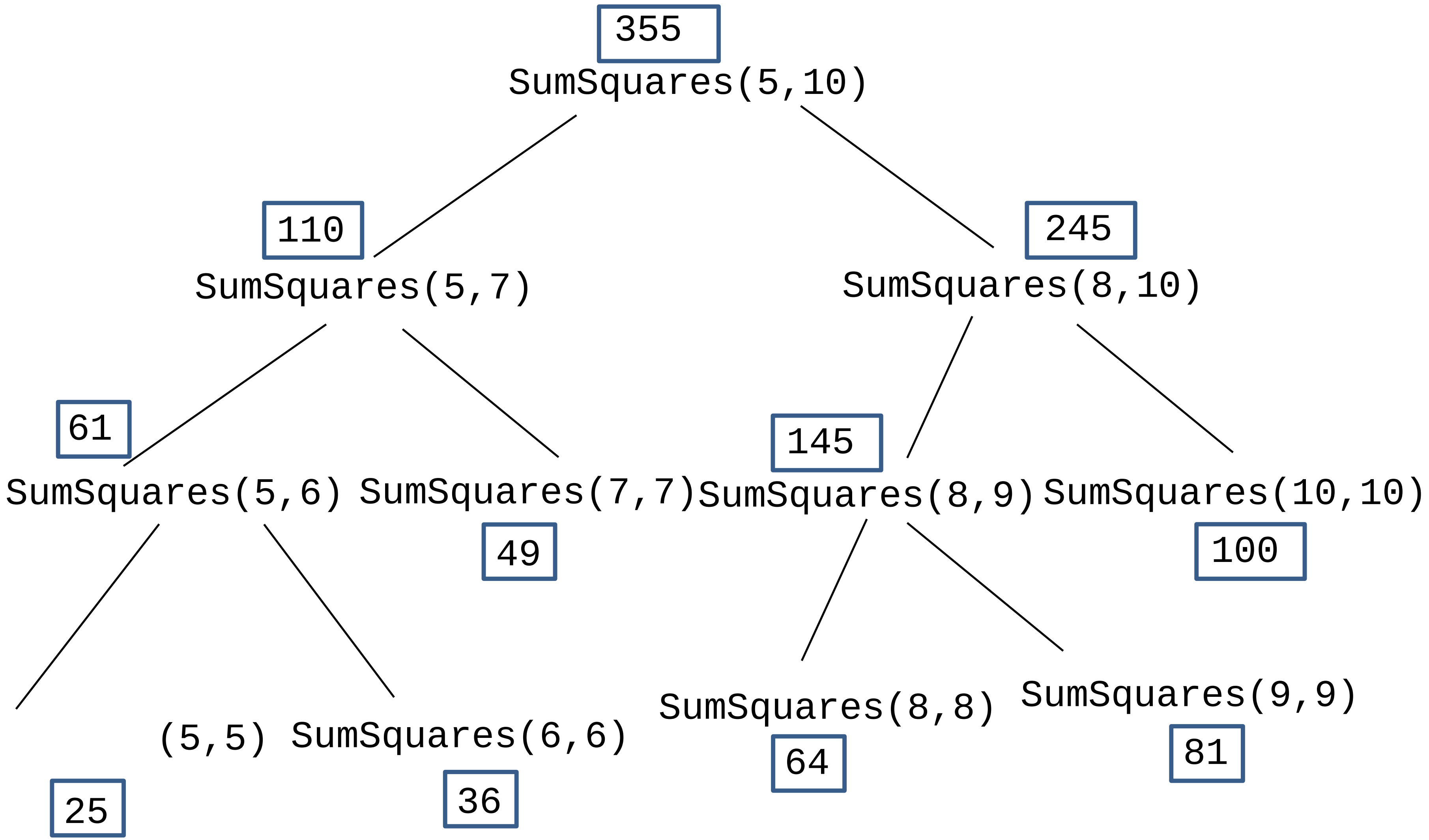
Call Trees and Traces

- We can depict graphically the behaviour of recursive programs by drawing **call trees** or **traces**.

Call Trees


SumSquares(5,10)
SumSquares(5,7)
SumSquares(8,10)
SumSquares(5,6)
SumSquares(7,7)
SumSquares(8,9)
SumSquares(10,10)
SumSquares(5,5)
SumSquares(6,6)
SumSquares(8,8)
SumSquares(9,9)

Annotated Call Trees



Traces

```
SumSquares(5,10) = SumSquares(5,7) + SumSquares(8,10)
= SumSquares(5,6) + SumSquares(7,7) + SumSquares(8,9) + SumSquares(10
= SumSquares(5,5) + SumSquares(6,6) + SumSquares(7,7) +
 + SumSquares(8,8) + SumSquares(9,9) + SumSquares(10,10)
= ((25 + 36) + 49)+((64 + 81) + 100)
= (61 + 49) + (145 + 100)
= (110 + 245)
= 355
```

Computing the Factorial

- Let us consider a simple program to compute the factorial n! of n.
- An **iterative function** to do this is the following:

```
int Factorial(int n) {
  int i, f;
  f = 1;
  for (i = 2; i <= n; i++)
    f = f * i;
return f;
}
```

Recursive Factorial

```
int Factorial(int n) {
  if (n == 1) {
    return 1;
  }
  else {
  return n * Factorial(n-1);
  }
}
```

Computing the Factorial (cont’d)

- The previous program is a “going-down” recursion.
- Can you write a “going-up” recursion for factorial?
- Can you write a recursion combining two half-solutions?
- The above tasks do not appear to be easy.

Computing the Factorial (cont'd)

- It is easier to first write a function Product(m,n) which **multiplies** together the numbers in the range m:n.
- Then $Factorial(n) = Product(1, n)$.

Multiplying m:n Together Using Half-Ranges

```
int Product(int m, int n) {
  int middle;
  if (m == n) {
  return m;
  }
  else {
  middle = (m + n) / 2;
  return Product(m,middle) * Product(middle + 1,n);
  }
}
```

Reversing Linked Lists

- Let us now consider the problem of reversing a linked list L.
- The type NodeType has been defined in the previous lecture as follows:

```c
typedef char AirportCode[4];
typedef struct NodeTag {
  AirportCode Airport;
  struct NodeTag * Link;
} NodeType;
```

Reversing a List Iteratively

- An **iterative function for reversing a list** is the following:

```
void Reverse(NodeType ** L){
  NodeType * R, * N, * L1;
  L1 = L1 * L;
  R = NULL;
  while (L1 != NULL) {
    N = L1;
    L1 = L1->Link;
    N->Link = R;
    R = N;
  }
  * L = R;
}
```

Question

- If in our main program we have a list with a pointer A to its first node, how do we call the previous function?

Answer

- We should make the following call:

```
Reverse(&A)
```

Reversing Linked Lists (cont'd)

- A recursive solution to the problem of reversing a list L is found by partitioning the list into its **head** $Head(L)$ and **tail** $Tail(L)$ and then concatenating the reverse of $Tail(L)$ with $Head(L)$.

Head and Tail of a List

- Let L be a list. $Head(L)$ is a list containing the first node of L. $Tail(L)$ is a list consisting of L's second and succeeding nodes.
- If $L == NULL$ then $Head(L)$ and $Tail(L)$ are not defined.
- If L consists of a single node then $Head(L)$ is the list that contains that node and $Tail(L)$ is $NULL$.

Example

- Let $L = (SAN, ORD, BRU, DUS)$. Then
- $Head(L) = (SAN)$ and
- $Tail(L) = (ORD, BRU, DUS)$.

Reversing Linked Lists (cont'd)

```
NodeType * Reverse(NodeType * L){
  NodeType * Head, * Tail;
  if (L == NULL) {
    return NULL;
  }
  else {
    Partition(L, &Head, &Tail);
    return Concat(Reverse(Tail), Head);
  }
}
```

Reversing Linked Lists (cont'd)

```
void Partition(NodeType * L, NodeType ** Head, NodeType ** Tail) {
  if (L != NULL) {
    * Tail = L->Link;
    * Head = L;
    (* Head)->Link = NULL;
  }
}
```

Reversing Linked Lists (cont'd)

```
NodeType *Concat(NodeType *L1, NodeType *L2) {
  NodeType * N;
  if (L1 == NULL) {
    return L2;
  }
  else {
    N = L1;
    while (N->Link != NULL)
      N = N->Link;
    N->Link = L2;
    return L1;
  }
}
```

Infinite Regress

Let us consider again the recursive factorial function:

```
int Factorial(int n){
  if (n == 1) {
    return 1;
  }
  else {
    return n * Factorial(n-1);
  }
}
```

What happens if we call $Factorial(0)$?

Infinite Regress (cont'd)

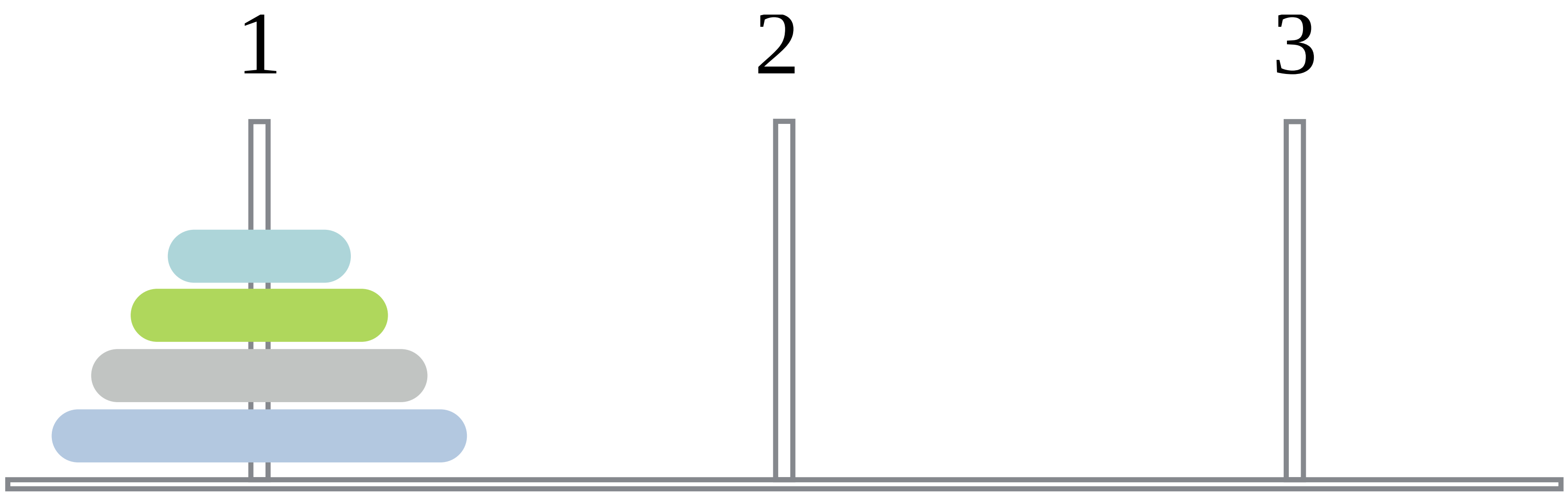
Factorial(0)= 0 * Factorial(-1)

= 0 * (-1) * Factorial(-2)

= 0 * (-1) * Factorial(-3)

- and so on, in an infinite regress.
- When we execute this function call, we get "Segmentation fault (core dumped)".

The Towers of Hanoi

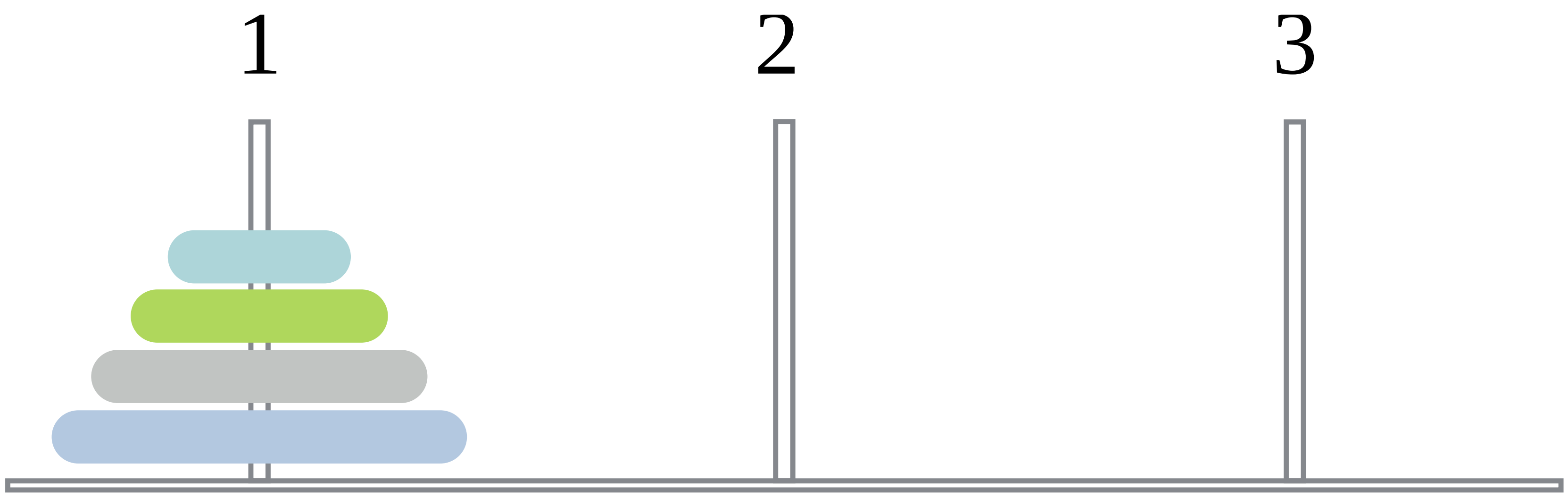


The Towers of Hanoi (cont'd)

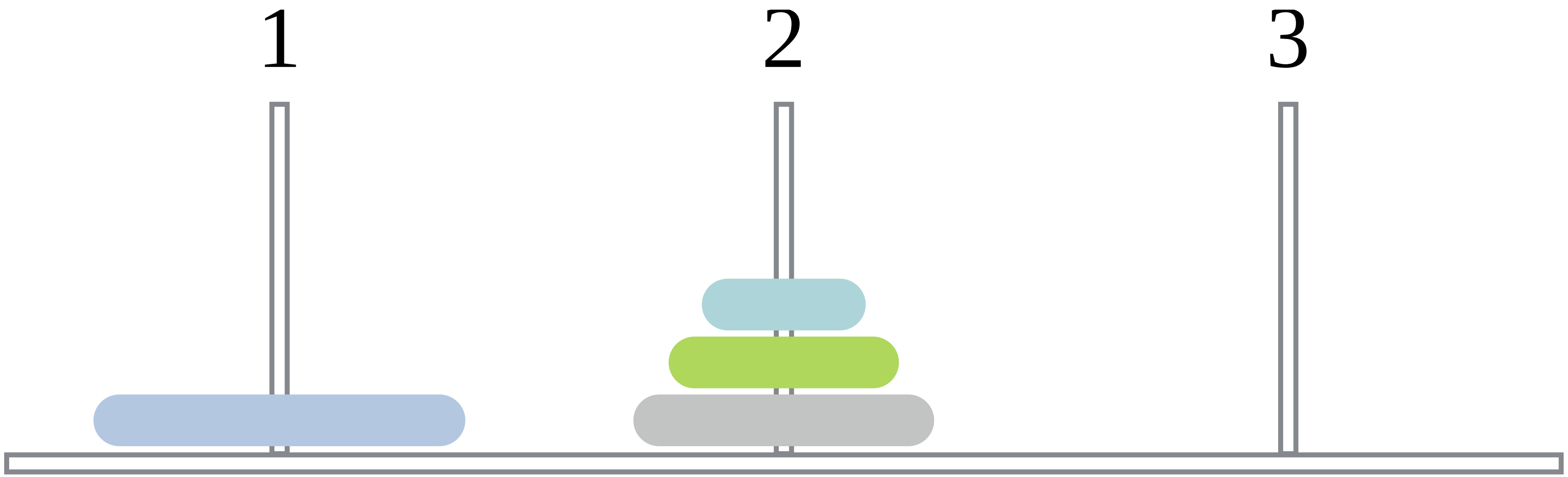
To Move 4 disks from Peg 1 to Peg 3:

- Move 3 disks from Peg 1 to Peg 2
- Move 1 disk from Peg 1 to Peg 3
- Move 3 disks from Peg 2 to Peg 3

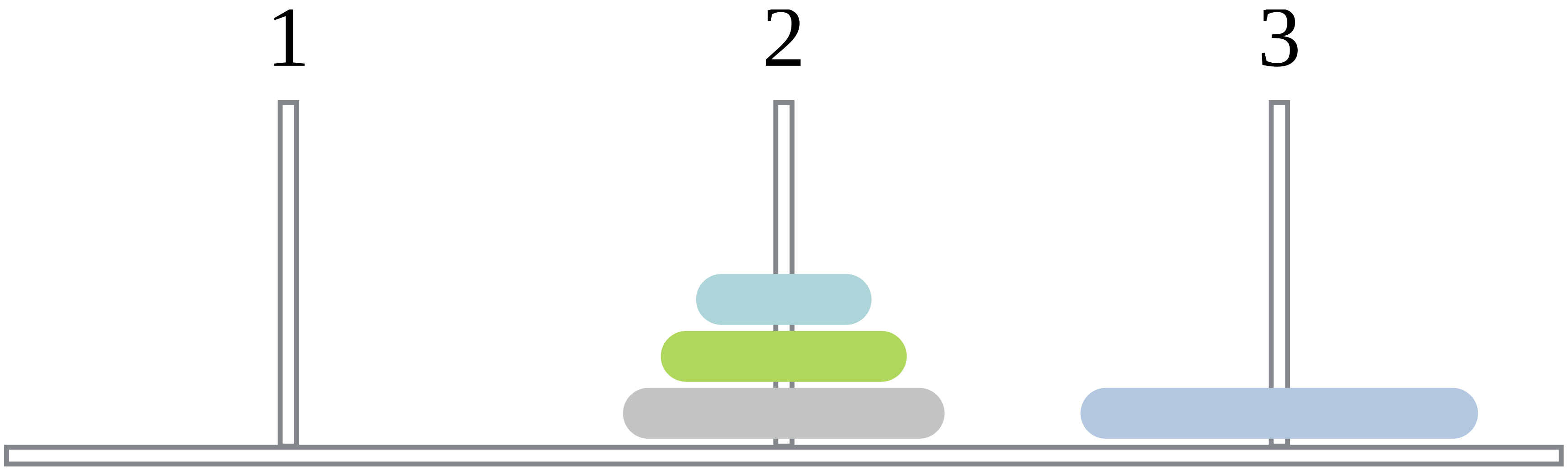
Move 3 Disks from Peg 1 to Peg 2



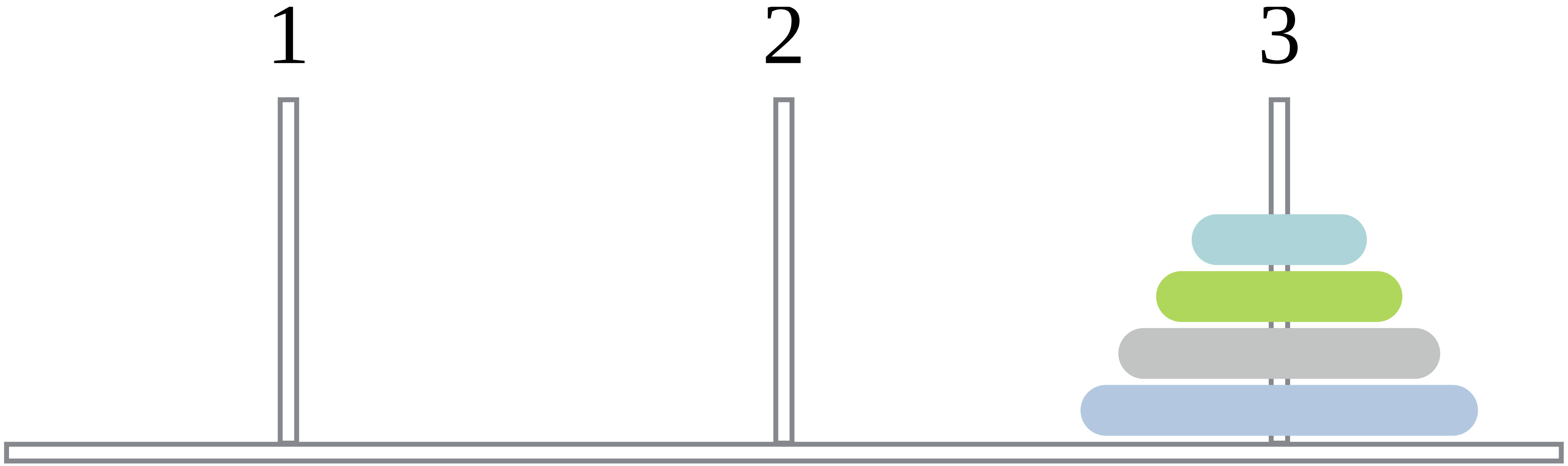
Move 1 Disk from Peg 1 to Peg 3



Move 3 Disks from Peg 2 to Peg 3



Done!



A Recursive Solution

```
void MoveTowers(int n, int start, int finish, int spare) {
  if (n == 1) {
    printf("Move a disk from peg %1d to peg %1d\n", start, finish);
  }
  else {
    MoveTowers(n-1, start, spare, finish);
    printf("Move a disk from peg %1d to peg %1d\n", start, finish);
    MoveTowers(n-1, spare, finish, start);
  }
}
```

Analysis

Let us now compute the **number of moves** L(n) that we need as a function of the number of disks n:

$$L(1) = 1 \, L(n) = L(n-1) + 1 + L(n-1) = 2 * L(n-1) + 1, n > 1$$

The above are called **recurrence relations**. They can be solved to give:

$$L(n) = 2n - 1$$

Analysis (cont'd)

- Techniques for solving recurrence relations are taught in the Algorithms and Complexity course.
- The running time of algorithm MoveTowers is **exponential** in the size of the input.

Readings

- T. A. Standish. *Data structures, algorithms and software principles in C*.
- Chapter 3.
- (προαιρετικά) R. Sedgewick. *Αλγόριθμοι σε C*. Κεφ. 5.1 και 5.2.