Weighted graphs

K08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού

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Weighted graphs

- Graphs with numbers, called **weights**, attached to each edge
  - Often restricted to **non-negative**
- Directed or undirected
- Examples
  - **Distance** between cities
  - **Cost** of flight between airports
  - **Time** to send a message between routers

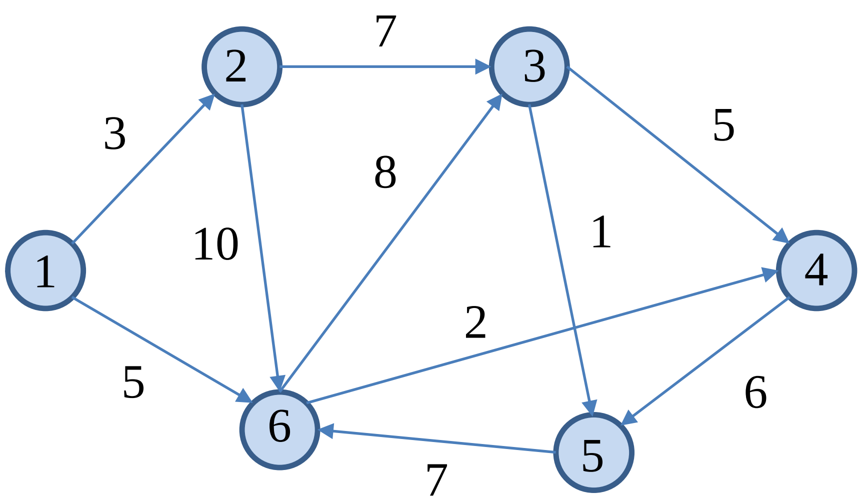
Weighted graphs

- Adjacency matrix representation

$$T[i,j] = \begin{cases} w_{i,j} & \text{if } i,j \text{ are connected} \\ \infty & \text{if } i \neq j \text{ are not connected} \\ 0 & \text{if } i = j \end{cases}$$

- Similarly for adjacency lists

Example weighted graph



Example weighted graph

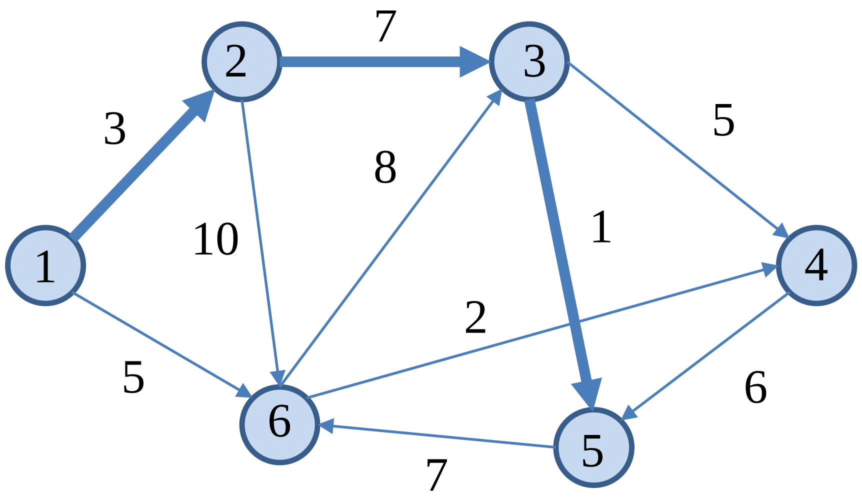
	1	2	3	4	5	6
1	0	3	∞	∞	∞	5
2	∞	0	7	∞	∞	10
3	∞	∞	0	5	1	∞
4	∞	∞	∞	0	6	∞
5	∞	∞	∞	∞	0	7
6	∞	∞	8	2	∞	0

Adjacency matrix

Shortest paths

- The **length** of a path is the **sum of the weights** of its edges
- Very common problem
  - find the **shortest path** from s to d
- Examples
  - Shortest route between cities
  - Cheapest connecting flight
  - Fastest network route
  - …

Shortest path from vertex 1 to vertex 5



Shortest path problem

Two main variants:

- **Single source** s
  - Find the shortest path from s to each node
  - **Dijkstra's** algorithm
    - Only for **non-negative** weights (important!)
- **All-pairs**
  - Find the shortest path between all pairs s, d
  - **Floyd-Warshall** algorithm
    - Any weights

Dijkstra's algorithm

Main ideas:

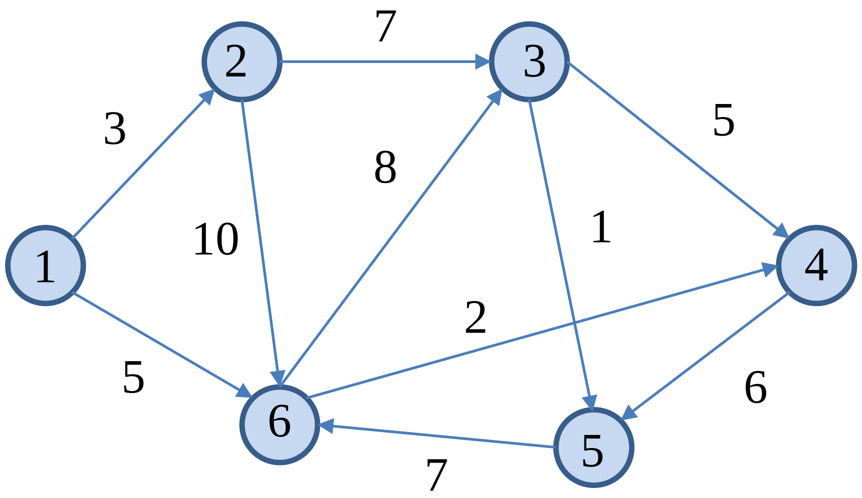
- Keep a set W of **visited** nodes
  - Start with $W = \{s\}$ (or $W = \{\}$)
- Keep a matrix $\Delta[u]$
  - Minimum distance from s to u **passing only through** W
  - Start with $\Delta[u] = T[s, u]$ (or $\Delta[s] = 0, \Delta[u] = \infty$)
- At each step we **enlarge** W by adding a **new vertex** $w \notin W$
  - w is the one with **minumum** $\Delta[w]$

Dijkstra's algorithm

Main ideas:

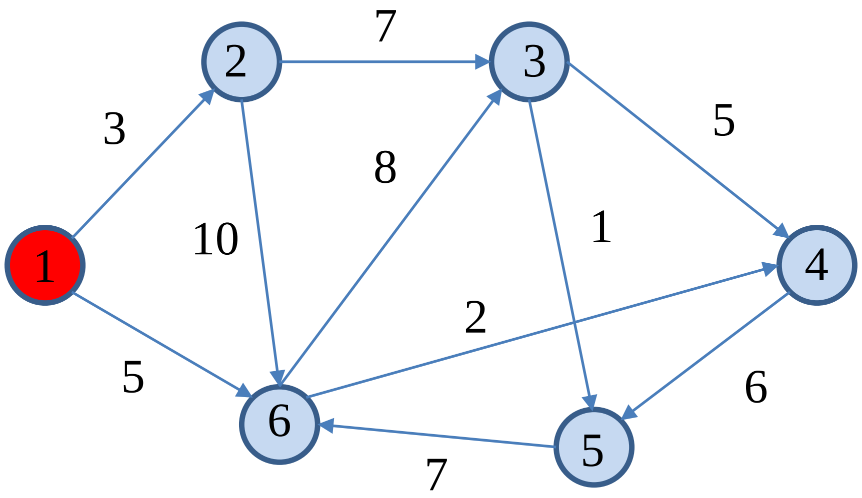
- Adding w to W might affect $\Delta[u]$
  - For some **neighbour** u of w
- We might now have a **shorter** path to u **passing through** w
  - Of the form $s \to \ldots \to w \to u$
  - If $\Delta[u] > \Delta[w] + T[w, u]$
- In this case we update Δ
  - $\Delta[u] = \Delta[w] + T[w, u]$

Example graph



Expanding the vertex set w in stages

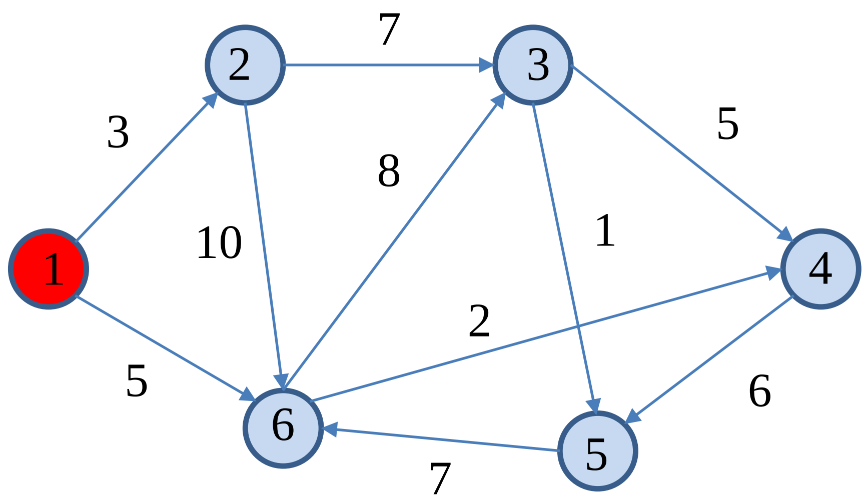
Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5



Expanding the vertex set w in stages

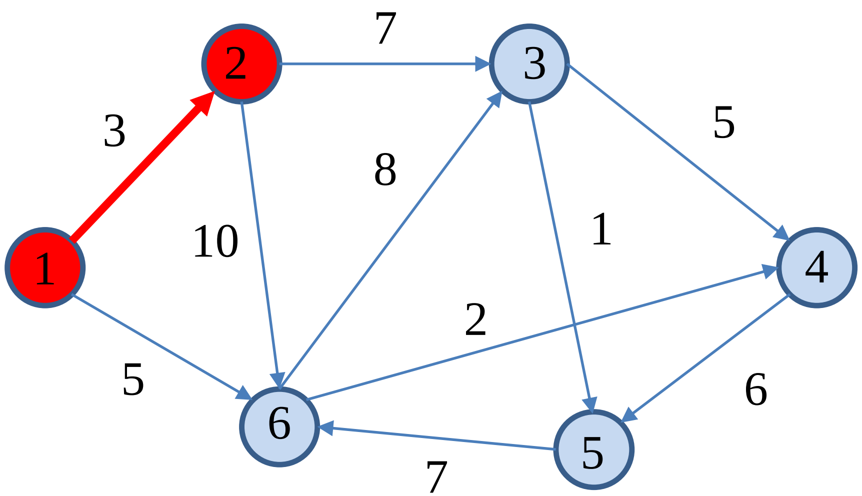
W=2 is chosen for the second stage.

Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5



Expanding the vertex set w in stages

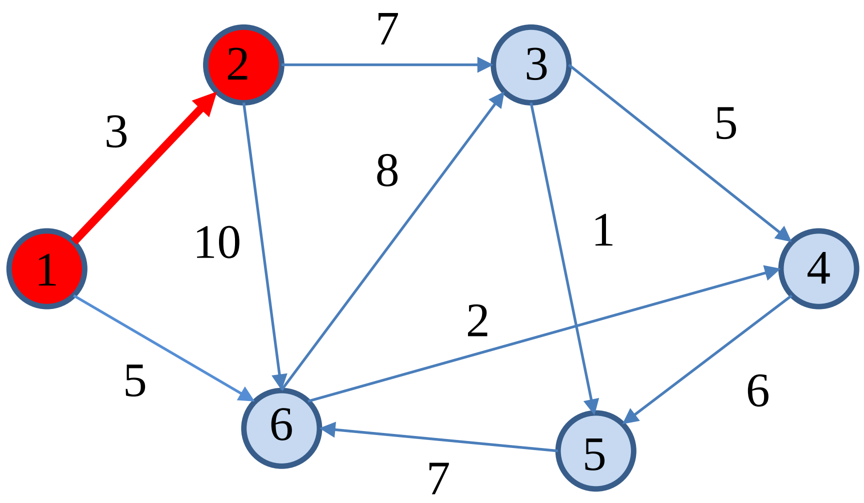
Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5



Expanding the vertex set w in stages

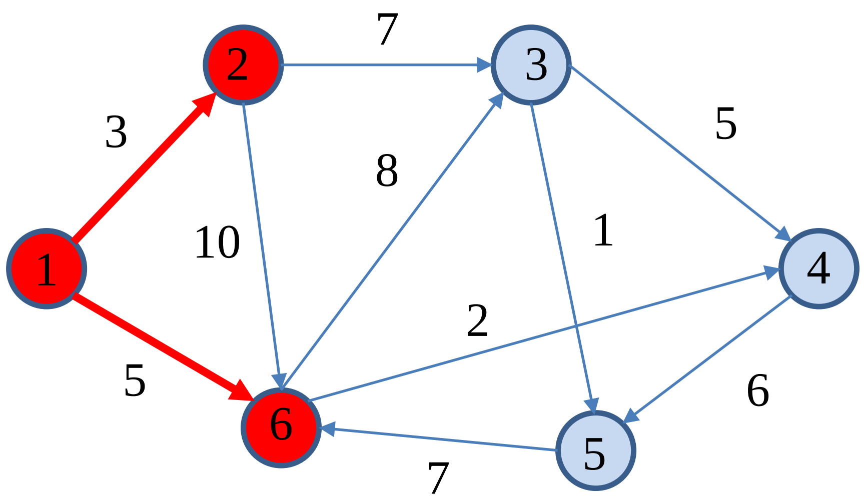
W=6 is chosen for the third stage.

Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5



Expanding the vertex set w in stages

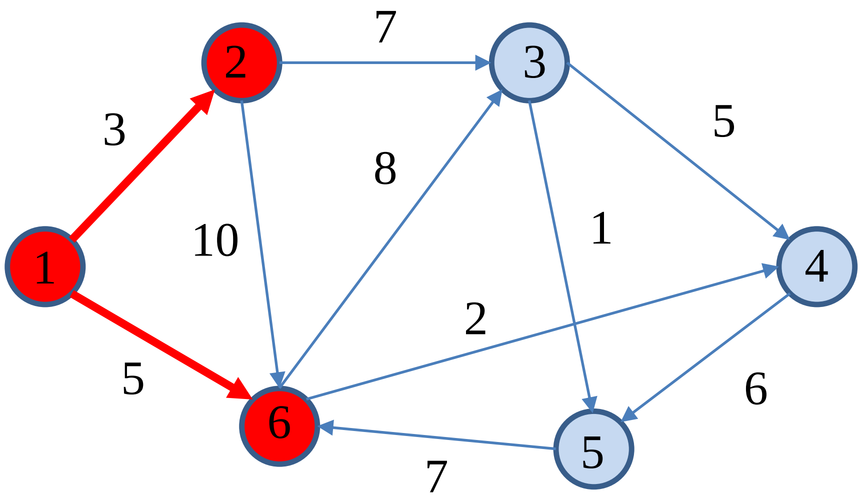
Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5
3	{1,2,6}	{3,4,5}	6	5	0	3	10	7	∞	5



Expanding the vertex set w in stages

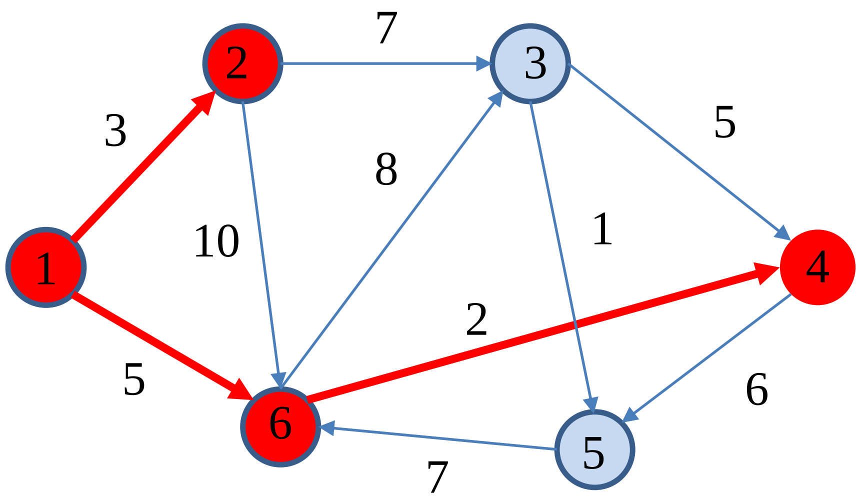
W=4 is chosen for the fourth stage.

Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5
3	{1,2,6}	{3,4,5}	6	5	0	3	10	7	∞	5



Expanding the vertex set w in stages

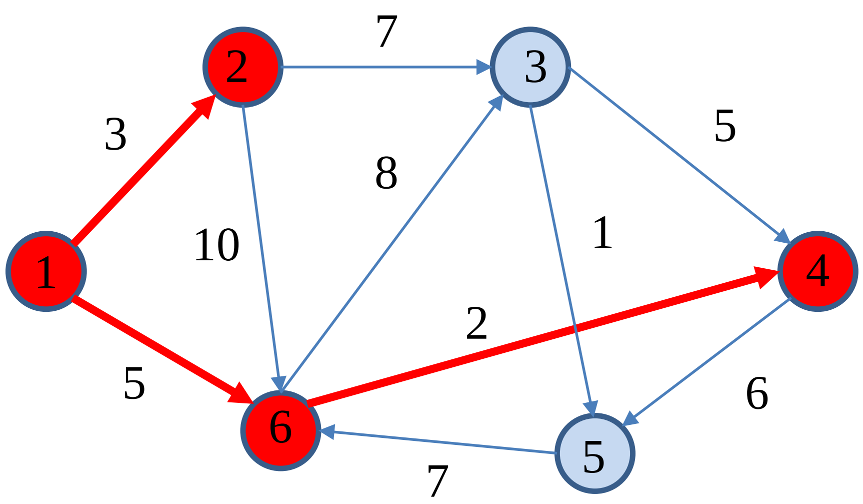
Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5
3	{1,2,6}	{3,4,5}	6	5	0	3	10	7	∞	5
4	{1,2,6,4}	{3,5}	4	7	0	3	10	7	13	5



Expanding the vertex set w in stages

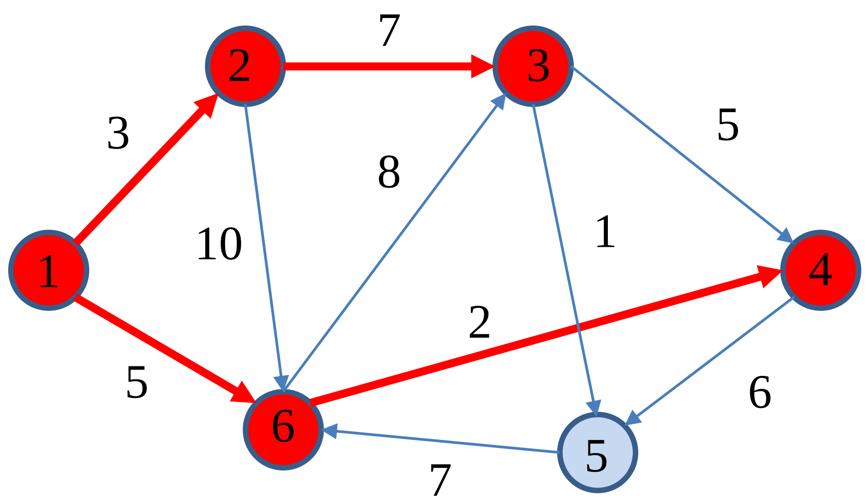
W=3 is chosen for the fifth stage.

Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5
3	{1,2,6}	{3,4,5}	6	5	0	3	10	7	∞	5
4	{1,2,6,4}	{3,5}	4	7	0	3	10	7	13	5



Expanding the vertex set w in stages

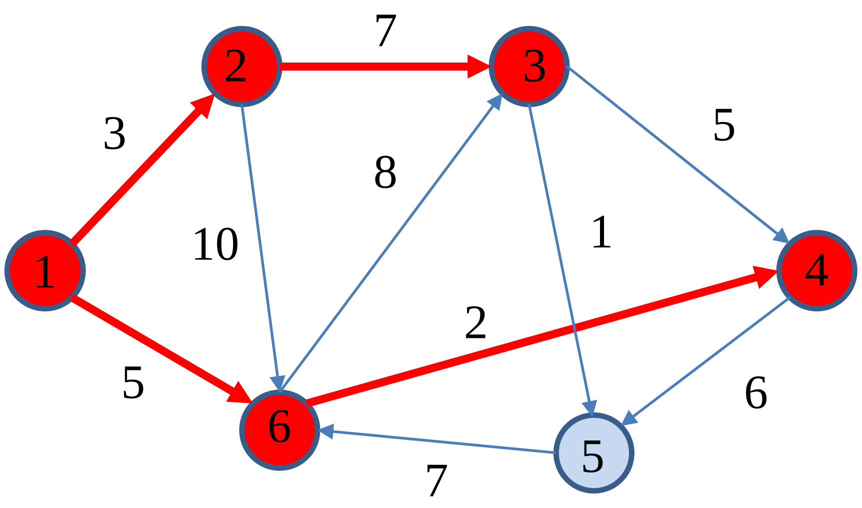
Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5
3	{1,2,6}	{3,4,5}	6	5	0	3	10	7	∞	5
4	{1,2,6,4}	{3,5}	4	7	0	3	10	7	13	5
5	{1,2,6,4,3}	{5}	3	10	0	3	10	7	11	5



Expanding the vertex set w in stages

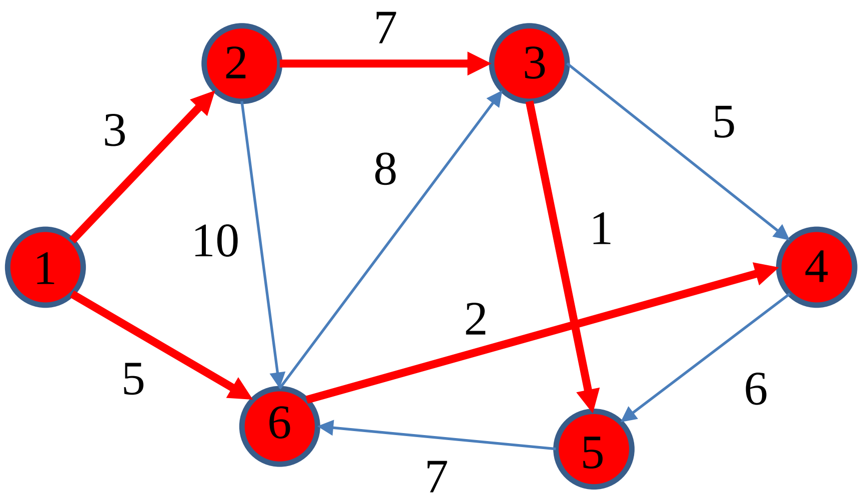
W=5 is chosen for the sixth stage.

Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5
3	{1,2,6}	{3,4,5}	6	5	0	3	10	7	∞	5
4	{1,2,6,4}	{3,5}	4	7	0	3	10	7	13	5
5	{1,2,6,4,3}	{5}	3	10	0	3	10	7	11	5



Expanding the vertex set w in stages

Stage	W	V-W	w	Δ(w)	Δ(1)	Δ(2)	Δ(3)	Δ(4)	Δ(5)	Δ(6)
Start	{1}	{2,3,4,5,6}	-	-	0	3	∞	∞	∞	5
2	{1,2}	{3,4,5,6}	2	3	0	3	10	∞	∞	5
3	{1,2,6}	{3,4,5}	6	5	0	3	10	7	∞	5
4	{1,2,6,4}	{3,5}	4	7	0	3	10	7	13	5
5	{1,2,6,4,3}	{5}	3	10	0	3	10	7	11	5
6	{1,2,6,4,3,5}	{}	5	11	0	3	10	7	11	5



Dijkstra’s algorithm in pseudocode

```
// Δεδομένα
src  : αρχικός κόμβος
dest : τελικός κόμβος

// Πληροφορίες που κρατάμε για κάθε κόμβο v
W[u]     : 1 αν ο u είναι στο σύνολο W, 0 διαφορετικά
dist[u]  : ο πίνακας Δ
prev[u]  : ο προηγούμενος του v στο βέλτιστο μονοπάτι

// Αρχικοποίηση: W={} (εναλλακτικά μπορούμε και W={src})
for each vertex u in Graph
  dist[u] = INT_MAX    // infinity
  prev[u] = NULL
  W[u] = 0

dist[src] = 0
```

Dijkstra’s algorithm in pseudocode

```
// Κυρίως αλγόριθμος
while true
    w = vertex with minimum dist[w], among those with W[w] = 0

    W[w] = 1
    if w == dest
        stop
        // optimal cost = dist[dest]
        // optimal path = dest <- prev[dest] <- ... <- src (inverse)

    for each neighbor u of w
        if W[u] == 1
            continue
        alt = dist[w] + weight(w,u)     // κόστος του src -> ... -> w
        if alt < dist[u]                // καλύτερο από πριν, update
            dist[u] = alt
            prev[u] = w
```

Using a priority queue

- Finding the $w \notin W$ with **minumum** $\Delta[w]$ is slow
  - $O(n)$ time
- But we can use a **priority queue** for this!
  - We only keep vertices $w \notin W$ in the queue
  - They are compared based on their $\Delta[w]$ (each w has "priority" $\Delta[w]$)
- Careful when $\Delta[w]$ is modified!
  - Either use a priority queue that allows **updates**
  - Or insert multiple copies of each w with **different priorities**
    - the queue might contain **already visited** vertices: ignore them

Dijkstra’s algorithm with priority queue

```
// Δεδομένα
src  : αρχικός κόμβος
dest : τελικός κόμβος

// Πληροφορίες που κρατάμε για κάθε κόμβο u
W[u]     : 1 αν ο v είναι στο σύνολο W, 0 διαφορετικά
dist[u]  : ο πίνακας Δ
prev[u]  : ο προηγούμενος του v στο βέλτιστο μονοπάτι
pq       : Priority queue, εισάγουμε tuples {u,dist[u]}
           συγκρίνονται με βάση το dist[u]

// Αρχικοποίηση: W={} (εναλλακτικά μπορούμε και W={src})
prev[src] = NULL
dist[src] = 0
pqueue_insert(pq, {src,0})  // dist[src] = 0
```

Dijkstra's algorithm with priority queue

```
// Κυρίως αλγόριθμος
while pq is not empty
    w = pqueue_max(pq)  // w with minimal dist[u]
    pqueue_remove_max(pq)

    if exists(W[w])     // το w μπορεί να υπάρχει πολλές φορές στην ο
        continue        // δεν κάνουμε replace), και να είναι ήδη vis
    W[w] = 1
    if w == dest
        stop            // optimal cost/path same as before

    for each neighbor u of w
        if exists(W[u])
            continue
        alt = dist[w] + weight(w,u)     // cost of src->...->w->u
        if !exists(dist[u]) OR alt < dist[u]
            dist[u] = alt
            prev[u] = w
            pqueue_insert(pq, {u,alt})  // προαιρετικά: replace αν υπ

stop // pq άδειασε πριν βρούμε το dest => δεν υπάρχει μονοπάτι
```

Notation

- $s \to d$
  - Direct step step from s to d
- $s \xrightarrow{W} d$
  - Multiple steps $s \to \ldots \to d$
  - All intermediate steps belong to the set $W \subseteq V$
- $s \xRightarrow{W} d$
  - Shortest path among all $s \xrightarrow{W} d$
  - So $s \xRightarrow{V} d$ is the overall shortest one

Proof of correctness

- We need to prove that $\Delta[u]$ is the **minimum distance to** u
  - **after** the algorithm finishes
- But it's much easier to reason **step by step**
  - we need a property that holds **at every step**
  - this is called an **invariant** (property that never changes)

Proof of correctness

Invariant of Dijkstra's algorithm

- $\Delta[u]$ is the cost of the shortest path **passing only through** W
- And the shortest **overall** when $u \in W$

Formally:

1. For all $u \in V$ the path $s \overset{W}{\Longrightarrow} u$ has cost $\Delta[u]$
2. For all $u \in W$ the path $s \overset{V}{\Longrightarrow} u$ has cost $\Delta[u]$

Proof: **induction** on the **size of** W, for both (1), (2) together

Proof of correctness

Base case $W = \{s\}$

- Trivial, the only path $s \xrightarrow{W} u$ is the direct one $s \to u$
- For (1): its cost is exactly $T[s,u] = \Delta[u]$
  - initial value of $\Delta[u]$
- For (2): the only $u \in W$ is s itself

Proof of correctness

Inductive case

- Assume $|W| = k$ and (1),(2) hold
- The algorithm
  - Updates W, adding a new vertex $w \notin W$
  - Updates $\Delta[u]$ for all neighbours u of w
- Let W', Δ' be the values **after** the update
- Show that (1),(2) still hold for W', Δ'

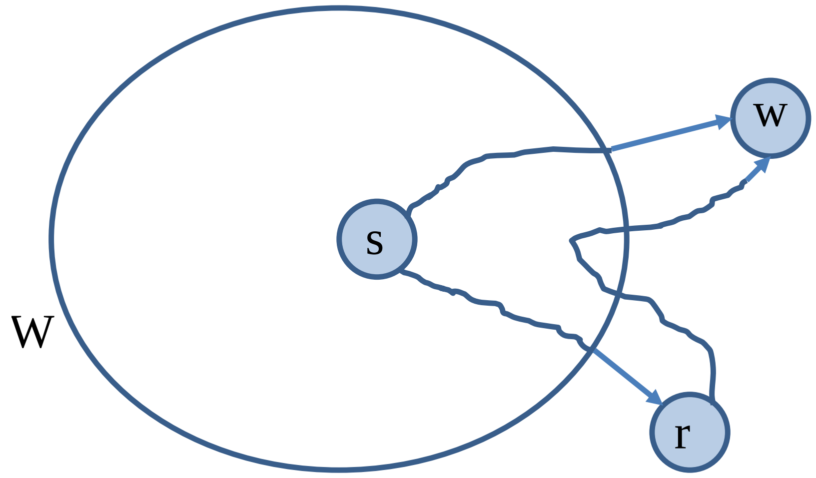
Proof of correctness

We start showing that (2) still holds for W', Δ'

- The interesting vertex is the w we just added
  - Vertices $u \neq w$ are trivial from the induction hypothesis
- Show: $s \overset{V}{\Longrightarrow} w$ has cost Δ'$[w]$
  - Note: Δ'$[w] = \Delta[w]$ (we do not update $\Delta[w]$)
  - We already know that $s \overset{W}{\Longrightarrow} w$ has cost $\Delta[w]$ (ind. hyp)
  - So we just need to prove that there is **no better** path **outside** W

Proof of correctness

- Assuming such path exists, let r be its **first** vertex outside W
  - So the path $s \overset{W}{\Longrightarrow} r \overset{V}{\Longrightarrow} w$ has cost $c < \Delta[w]$
  - So the path $s \overset{W}{\Longrightarrow} r$ has cost at most $c < \Delta[w]$ (no negative weights!)
  - So $\Delta[r] < \Delta[w]$
- **Impossible!** We chose w to be the one with min $\Delta[w]$



Proof of correctness

It remains to show (1) for W', Δ'

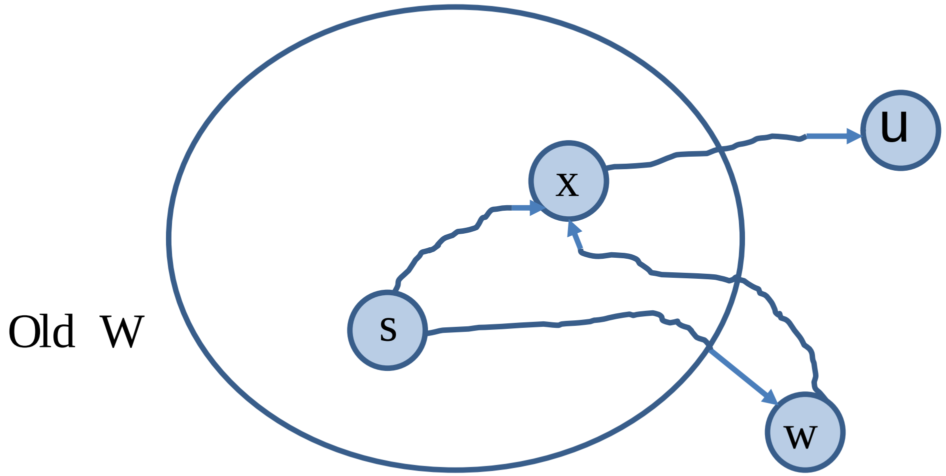
- Take some arbitrary u
  - Let c be the cost of $s \overset{W'}{\Longrightarrow} u$
  - Show: $c = \Delta'[u]$
- Three cases for the optimal path $s \overset{W'}{\Longrightarrow} u$
- Case 1: the path does not pass through w
  - So it is of the form $s \overset{W}{\Longrightarrow} u$
  - This path has cost $\Delta[u]$ (induction hypothesis)
  - No update: $\Delta'[u] = \Delta[u] = c$

Proof of correctness

- Case 2: w is right before u
  - So the path is of the form $s \xRightarrow{W} w \to u$
  - The cost of $s \xRightarrow{W} w$ is $\Delta[w]$ (induction hypothesis)
  - So $c = \Delta[w] + T[w, u]$
  - So the algorithm will set $\Delta'[u] = \Delta[w] + T[w, u]$ when updating the neighbours of w
  - So $c = \Delta'[u]$

Proof of correctness

- Case 3: some other x appears after w in the path
  - So the path is of the form $s \overset{W}{\Longrightarrow} w \rightarrow x \overset{W}{\Longrightarrow} u$
  - But the path $s \overset{W}{\Longrightarrow} w \rightarrow x$ is no shorter than $s \overset{W}{\Longrightarrow} x$
    - From the induction hypothesis for $x \in W$
  - So $s \overset{W}{\Longrightarrow} x \rightarrow u$ is also optimal, reducing to case 1!



Complexity

Without a priority queue:

- Initialization stage: loop over vectices: $O(n)$
- The while-loop adds one vertex every time: n iterations
- Finding the new vertex loops over vertices: $O(n)$
  - same for updating the neighbours
- So total $O(n^2)$ time

Complexity

With a priority queue:

- Initialization stage: loop over vectices, so $O(n)$
- Count the number of **updates** (steps in the **inner** loop)
  - Once for every neighbour of every node: e total
  - Each update is $O(\log n)$ (at most n elements in the queue)
- Total $O(e \log n)$
  - Assuming a connected graph ($e \geq n$)
  - And an implementation using adjacency lists
- Only good for **sparse** graphs!
  - But $O(n \log n)$ can be hugely better than $O(n^2)$

The all-pairs shortest path problem

- Find the shortest path between all pairs s, d
- **Floyd-Warshall** algorithm
- Any weights
  - Even negative
  - But no **negative loops** (why?)

The all-pairs shortest path problem

Main idea

- At each step we compute the shortest path through a **subset of vertices**
  - Similarly to W in Dijkstra
  - But now the set at step k is $W_k = \{1, \ldots, k\}$
    - Vectices are numbered in any order
- Step k: the cost of $i \stackrel{W_k}{\Longrightarrow} j$ is $A_k[i, j]$
  - Similar to Δ in Dijstra (but for all **pairs** i, j of nodes)

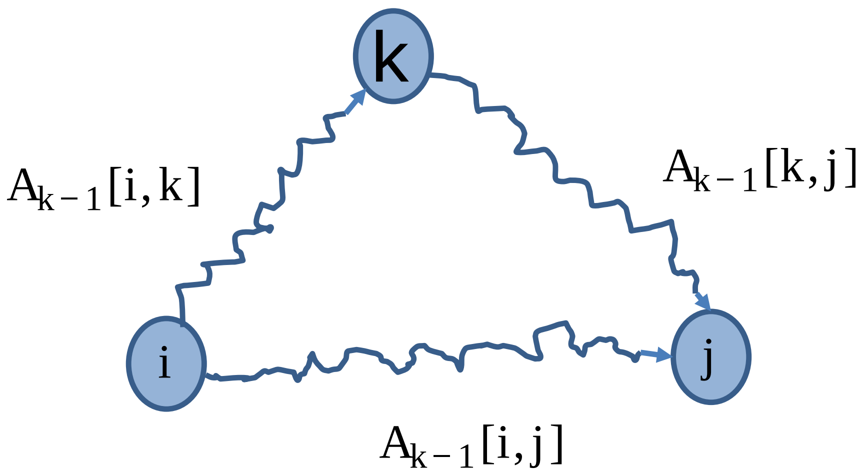
Floyd-Warshall algorithm

- The algorithm at each step computes A_k from A_{k-1}
- Initial step $k = 0$
  - Start with $A_0[i, j] = T[i, j]$
  - Only direct paths are allowed

Floyd-Warshall algorithm

k-th iteration: the optimal $i \xrightarrow{W_k} j$ either **passes thorugh** k or not.

$$A_k[i,j] = \min \begin{cases} A_{k-1}[i,j] \\ A_{k-1}[i,k] + A_{k-1}[k,j] \end{cases}$$



Floyd-Warshall algorithm in pseudocode

```
void floyd_warshall() {

    for (int i = 0; i <= size-1; i++)
        for (int j = 0; j <= size-1; j++)
            A[i][j] = weight(i, j)

    for (int i = 0; i <= size-1; i++)
        A[i][i] = 0;

    for (int k = 0; k <= size-1; k++)
        // Compute A_k from A_{k-1}
        for (int i = 0; i <= size-1; i++)
            for (int j = 0; j <= size-1; j++)
                if (A[i][k] + A[k][j] < A[i][j])
                    A[i][j] = A[i][k] + A[k][j]
}
```

`A` is the **current** A_k at every step k.

Complexity

- Three simple loops of n steps
- So $O(n^3)$
- **Not** better than n executions of Dijkstra in **complexity**
  - But much simpler
  - Often faster in practice
  - And works for **negative** weights

Readings

- T. A. Standish. *Data Structures , Algorithms and Software Principles in C.* Chapter 10
- A. V. Aho, J. E. Hopcroft and J. D. Ullman. *Data Structures and Algorithms.* Chapters 6 and 7